Instanton calculus in deformed $\mathcal{N} = 4$ super Yang-Mills theories

Katsushi Ito,[a] **Hiroaki Nakajima,**[b] **Takuya Saka**[a] **and Shin Sasaki**[a]

[a]*Department of Physics, Tokyo Institute of Technology,*
Tokyo, 152-8551, Japan

[b]*BK21 Physics Research Division and Institute of Basic Science, Sungkyunkwan University,*
Suwon, 440-746, Korea

E-mail: ito@th.phys.titech.ac.jp, nakajima@skku.edu,
bstsaka@th.phys.titech.ac.jp, shin-s@th.phys.titech.ac.jp

ABSTRACT: We study the instanton effective action of four-dimensional deformed $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the presence of constant, self-dual Ramond-Ramond (R-R) 3-form background in type IIB superstring theory. We compare the instanton effective action with the low-energy effective action on D(-1)-branes in the D3-D(-1) system in the same background. We find that discrepancy appears at the second order of the R-R background, which was also observed in deformed $\mathcal{N} = 2$ theory. This discrepancy is resolved if the action of the deformed gauge theory is improved by introducing a term with coordinate-dependent gauge coupling. We obtain the same deformed instanton effective action from super Yang-Mills theory in ten-dimensional Ω-background by dimensional reduction. We also discuss another type of R-R 3-form background which corresponds to massive deformations of the instanton effective action.

Contents

1 Introduction

It has been known that closed string backgrounds induce non-trivial effects on D-branes, which are useful to investigate non-perturbative effects in gauge theories. The low-energy effective field theories on D-branes in closed string backgrounds are described by deformed gauge theories. For example, the field theory on D-branes in the constant NS-NS B-field background is described by noncommutative gauge theory [1, 2]. It was shown that non-commutativity resolves the small instanton singularity in the moduli space of instantons [3].

Since superstring theory contains various Ramond-Ramond (R-R) closed string back-grounds, it would be interesting to study supersymmetric gauge theories deformed in R-R backgrounds. These deformed theories are useful to investigate non-perturbative effects in field theories and also stringy effects from the viewpoint of field theory. Constant self-dual graviphoton background, for example, plays an important role to study F-terms in $\mathcal{N} = 1$ supersymmetric gauge theories [4–7]. This background originates from the self-dual R-R 5-form in type IIB superstring theory.

To obtain deformed low-energy effective theories in closed string backgrounds by taking the zero-slope limit $\alpha' \to 0$, we need to specify the scaling condition for the backgrounds. For example, the constant self-dual graviphoton background $\mathcal{F}^{(\alpha\beta)}$, where α, β are four-dimensional spinor indices and $(\alpha\beta)$ denotes the symmetrization with respect to α and β, with fixed $(2\pi\alpha')^{\frac{3}{2}}\mathcal{F}^{(\alpha\beta)}$ induces non(anti)commutativity in $\mathcal{N} = 1$ superspace [8, 9]. Supersymmetric gauge theories and deformed instantons in the $\mathcal{N} = 1$ non(anti)commutative

superspace have been studied extensively [10–12], which are realized by D3-branes and D(-1)-branes at the singularity of the orbifold $\mathbf{C}^3/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ in the graviphoton background [13].

When we consider D3-branes at the $\mathbf{C}^2/\mathbf{Z}_2$ orbifold singularity, the low-energy effective theory is described by $\mathcal{N} = 2$ super Yang-Mills theory. The self-dual R-R 5-form background $\mathcal{F}^{(\alpha\beta)(IJ)}$ with the scaling condition $(2\pi\alpha')^{\frac{3}{2}}\mathcal{F}^{(\alpha\beta)(IJ)} = $ fixed, where $I, J = 1, 2$ are $\mathrm{SU}(2)_R$ R-symmetry indices, induces $\mathcal{N} = 2$ non(anti)commutative superspace with non-singlet deformation [14, 15]. The constant R-R 1-form field strength background $\mathcal{F}^{[\alpha\beta][IJ]}$, where $[\alpha\beta]$ denotes the anti-symmetrization with respect to α and β, is expected to provide the singlet deformation (see for example [16]). The low-energy effective theory of D3-branes in the self-dual R-R 5-form background $\mathcal{F}^{(\alpha\beta)(AB)}$ $(A, B = 1, \ldots, 4)$ on the flat ten-dimensional spacetime has been studied in [17, 18], which would correspond to non(anti)commutative $\mathcal{N} = 4$ super Yang-Mills theory [19–21]. Here A and B are $\mathrm{SU}(4)_R$ R-symmetry indices.

Recently, low-energy effective theories on D-branes in the constant R-R 3-form field strength background have been attracted much attentions in study of non-perturbative effects in supersymmetric gauge theories. There are two types of the constant R-R 3-form field strength: $\mathcal{F}^{(\alpha\beta)[IJ]}$ and $\mathcal{F}^{[\alpha\beta](IJ)}$. We call these (S,A) and (A,S)-type background, respectively. In [22], they studied the low-energy effective action of D(-1)-branes in the D3-D(-1) system located at the singularity of the orbifold $\mathbf{C}^2/\mathbf{Z}_2$ in the (S,A)-background with the scaling condition $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[IJ]} = $ fixed and found that the action coincides with the instanton effective action of $\mathcal{N} = 2$ super Yang-Mills theory in the Ω-background. This deformed instanton effective action is very useful to obtain the exact formula of the prepotential via localization technique [23, 24].

In [25], we studied the deformed $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super Yang-Mills theories as the low-energy effective action of the D3-branes in the (S,A) and (A,S)-type backgrounds with the same scaling condition as in [22]. Then it is natural to expect that the instanton effective action derived from the D3-D(-1) system can be also calculated from the ADHM construction of instantons [26] of the deformed gauge theory. However, in [27, 28] we found that there exists a discrepancy between the ADHM construction of instanton effective action in deformed $\mathcal{N} = 2$ super Yang-Mills theory and the D(-1)-brane effective action in the D3-D(-1) system at the second order in the R-R background. This is due to the fact that the (S,A)-background field gives the mass term to the moduli corresponding to the position of D(-1)-branes, which breaks the translational symmetry. We showed that if we improve the action of the deformed gauge theory by adding a term with coordinate-dependent gauge coupling constant, we can recover the instanton effective action obtained from the D3-D(-1) system. Although we do not know yet string theoretical derivation of this improvement term, the deformed gauge theory in the (S,A)-background provides a simple method to obtain the deformed instanton effective action from the D-brane configuration. Therefore it is interesting to investigate whether this kind of improvement of the deformed action happens in other theories.

In this paper, we will generalize the $\mathcal{N} = 2$ calculations [27, 28] to the case of deformed $\mathcal{N} = 4$ super Yang-Mills theories. Since $\mathcal{N} = 4$ theories have maximal supersymmetry, the

(S,A) and (A,S)-type backgrounds admit more general deformations and the deformed $\mathcal{N} = 2$ theories can be obtained by the orbifold projection. The deformed effective action of the D(-1)-branes embedded in the D3-branes is evaluated by the open string disk amplitudes with background corrections and is compared with the ADHM instanton calculus based on the deformed D3-brane action. As in the case of the $\mathcal{N} = 2$ theory, we find a disagreement between them at the second order in the deformation parameter. This disagreement can be resolved by adding a term with spacetime coordinate-dependent gauge coupling, which we call the improvement of the action. We then find that both approaches give the same result. We also find that this instanton effective action is obtained from $\mathcal{N} = 4$ gauge theory in the Ω-background defined in ten dimensions, which is a natural generalization of the six-dimensional Ω-background in $\mathcal{N} = 2$ theory [23].

We will also discuss the (A,S)-type deformation of the instanton effective action. In the case of the (S,A)-type deformation, certain bosonic ADHM moduli have mass term, whose mass depends on the deformation parameters. In the case of the (A,S)-type, the chiral fermions in deformed super Yang-Mills theory have mass term. In fact, the (A,S)-deformed instanton effective action becomes that of mass deformed $\mathcal{N} = 4$ super Yang-Mills theories. In the instanton effective action, this corresponds to mass term for certain fermionic moduli.

The organization of this paper is as follows. In the next section, we evaluate open string disk amplitudes corresponding to the string that at least one of the end points is attached on the D(-1)-branes and calculate the deformed instanton effective action up to the second order in the deformation parameter. In section 3, we study the (S,A)-type deformed $\mathcal{N} = 4$ super Yang-Mills theory on the D3-branes and the ADHM construction of instantons. In section 4, we study the Ω-background deformation of $\mathcal{N} = 4$ super Yang-Mills theory and compare the results with those obtained in section 2 and 3. In section 5, we study the deformation in the (A,S)-type background. Section 6 is devoted to conclusions and discussion. Detailed calculations of open string disk amplitudes are presented in appendix A. A brief introduction of the ADHM construction can be found in appendix B.

2   (S,A)-deformed $\mathcal{N} = 4$ instanton effective action from string amplitudes

In this section we calculate the instanton effective action based on the low-energy effective action of the D3-D(-1) system deformed in the (S,A)-type background. Firstly we summarize notations and conventions used in this paper. We denote $\alpha, \dot{\alpha} = 1, 2$ spinor indices of four dimensional Euclidean spacetime $x^m = (x^1, x^2, x^3, x^4)$, which is the world-volume of D3-branes. We follow the notation of [29]. Euclidean sigma matrices are $\sigma_{m\alpha\dot{\alpha}} = (i\tau^1, i\tau^2, i\tau^3, 1)$ and $\bar{\sigma}_m^{\dot{\alpha}\alpha} = (-i\tau^1, -i\tau^2, -i\tau^3, 1)$, where τ^c ($c = 1, 2, 3$) are the Pauli matrices. They satisfy $\sigma^m\bar{\sigma}^n + \sigma^n\bar{\sigma}^m = \bar{\sigma}^m\sigma^n + \bar{\sigma}^n\sigma^m = 2\delta^{mn}$. The Lorentz generators are $\sigma_{mn} = \frac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m)$ and $\bar{\sigma}_{mn} = \frac{1}{4}(\bar{\sigma}_m\sigma_n - \bar{\sigma}_n\sigma_m)$. The 't Hooft symbols η^c_{mn} and $\bar{\eta}^c_{mn}$ are defined by $\sigma_{mn} = \frac{i}{2}\eta^c_{mn}\tau^c$ and $\bar{\sigma}_{mn} = \frac{i}{2}\bar{\eta}^c_{mn}\tau^c$. We use $A = 1, \ldots, 4$ for $SU(4)_R$ R-symmetry indices for the rotation in six-dimensional space $x^a = (x^5, \ldots, x^{10})$, which represents transverse directions of D3-branes. The six-dimensional matrices Σ^a and

$\bar{\Sigma}^a$ $(a = 5, \ldots, 10)$ are

$$(\Sigma^a)_{AB} = \left(\eta^3, -i\bar{\eta}^3, \eta^2, -i\bar{\eta}^2, \eta^1, i\bar{\eta}^1\right), \quad (\bar{\Sigma}^a)^{AB} = (-\eta^3, -i\bar{\eta}^3, -\eta^2, -i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1), \ (2.1)$$

which satisfy $\bar{\Sigma}^a \Sigma^b + \bar{\Sigma}^b \Sigma^a = \Sigma^b \bar{\Sigma}^a + \Sigma^a \bar{\Sigma}^b = 2\delta^{ab}$. The Lorentz generators are $\Sigma^{ab} = \frac{1}{4}(\Sigma^a \bar{\Sigma}^b - \Sigma^b \bar{\Sigma}^a)$ and $\bar{\Sigma}^{ab} = \frac{1}{4}(\bar{\Sigma}^a \Sigma^b - \bar{\Sigma}^b \Sigma^a)$.

Instantons with topological number k are obtained by the ADHM construction and are parametrized by the ADHM moduli [26]. In $\mathcal{N} = 4$ super Yang-Mills theory, the bosonic ADHM moduli are $(a'_{\alpha\dot{\alpha}})_{ij}$ and $w_{uj\dot{\alpha}}$, where $i, j = 1, \ldots, k$ are instanton indices, $u = 1, \ldots, N$ is a gauge index. We define $(a'_m)_{ij}$ as $(a'_m)_{ij} = \frac{1}{2}\bar{\sigma}_m^{\dot{\alpha}\alpha}(a'_{\alpha\dot{\alpha}})_{ij}$. In addition, we introduce bosonic auxiliary fields χ_a and D^c $(c = 1, 2, 3)$, which are $k \times k$ complex matrices. D^c is the Lagrange multiplier for the ADHM constraints. Fermionic ADHM moduli are μ_{uj} and $(\mathcal{M}'^A_{\alpha})_{ij}$. There exist fermionic auxiliary fields $\bar{\psi}^{\dot{\alpha}}_A$ for the fermionic ADHM constraints.

In string theory, gauge theory instantons with instanton number k are understood as k D(-1)-branes embedded in D3-brane worldvolume [30]. The $\mathcal{N} = 4$ ADHM moduli parameters of the instantons correspond to the zero-modes of the open strings whose at least one endpoint is attached to the D(-1)-branes. The instanton effective action is obtained by evaluating the disk amplitudes which contain vertex operators associated with the ADHM moduli and the vacuum expectation values (VEVs) of the adjoint scalars living on the D3-branes. The effect of closed string backgrounds is interpreted as the insertions of the closed string vertex operators in the disk [22].

We start from reviewing the undeformed instanton effective action in the absence of the (S,A)-type background [31]. We use the NSR formalism to calculate disk amplitudes. The relevant vertex operators associated with the instanton moduli are listed in table 1 [31]. Here we denote the vertex operator for moduli Φ in the q-picture by $V^{(q)}_\Phi$. The worldsheet fermions are decomposed into ψ^m and ψ^a. ϕ is the bosonized ghost whose momentum in a vertex operator specifies the picture number. Δ and $\bar{\Delta}$ are twist fields, which interchange the D3 and D(-1) boundary in a disk amplitude [32]. The ten-dimensional spin field is decomposed into the four-dimensional part S^α, $S_{\dot{\alpha}}$ and the six-dimensional part S^A, S_A. More detailed explanation for the convention and notation of these worldsheet variables can be found in [14, 17]. The zero-modes from the D(-1)/D(-1) strings belong to the adjoint representation of U(k). Its generators t^U are normalized as $\text{tr}_k[t^U t^V] = \kappa \delta^{\text{UV}}$, where the trace is taken over $k \times k$ matrices. $g_0 = (2\pi)^{-\frac{3}{2}} g_s^{\frac{1}{2}} \alpha'^{-1}$ is the coupling constant of the D(-1)-branes [33], which should go to infinity in the zero-slope limit with fixed string coupling constant g_s. Note that in order to reproduce the correct field theory result, some of the moduli should be rescaled by g_0 in the zero-slope limit [31]. In addition to the $\mathcal{N} = 4$ ADHM moduli fields it is convenient to introduce new auxiliary fields $Y_{ma}, X_{\dot{\alpha}a}, \overline{X}^{\dot{\alpha}}{}_a$ to write quartic interaction terms from the cubic interactions for the auxiliary fields as in the case of $\mathcal{N} = 2$ [22].

From the charge conservation for the bosonized fermions in the correlator, it is easy

Brane	Vertex Operator	Representation
D(−1)/D(−1)	$V_{a'}^{(-1)}(y) = \dfrac{\pi}{\sqrt{2}}(2\pi\alpha')^{\frac{1}{2}}g_0 a'_m \psi^m e^{-\phi}(y)$	
	$V_{\chi}^{(-1)}(y) = \dfrac{1}{\sqrt{2}}(2\pi\alpha')^{\frac{1}{2}}\chi_a \psi^a e^{-\phi}(y)$	
	$V_{\mathcal{M}}^{(-1/2)}(y) = \pi(2\pi\alpha')^{\frac{3}{4}}g_0 \mathcal{M}'^{\alpha A}S_\alpha S_A e^{-\frac{1}{2}\phi}(y)$	U(k) adjoint
	$V_{\bar\psi}^{(-1/2)}(y) = 2(2\pi\alpha')^{\frac{3}{4}}\bar\psi_{\dot\alpha A}S^{\dot\alpha}S^A e^{-\frac{1}{2}\phi}(y)$	
	$V_D^{(0)}(y) = 2(2\pi\alpha')D_c \bar\eta^c_{mn}\psi^n \psi^m(y)$	
	$V_Y^{(0)}(y) = 4\pi(2\pi\alpha')g_0 Y_{ma}\psi^m \psi^a(y)$	
D3/D(−1)	$V_w^{(-1)}(y) = \dfrac{\pi}{2}(2\pi\alpha')^{\frac{1}{2}}g_0 w_{\dot\alpha}\Delta S^{\dot\alpha}e^{-\phi}(y)$	
	$V_{\bar w}^{(-1)}(y) = \dfrac{\pi}{2}(2\pi\alpha')^{\frac{1}{2}}g_0 \bar w_{\dot\alpha}\overline\Delta S^{\dot\alpha}e^{-\phi}(y)$	
	$V_\mu^{(-1/2)}(y) = \pi(2\pi\alpha')^{\frac{3}{4}}g_0 \mu^A \Delta S_A e^{-\frac{1}{2}\phi}(y)$	U(k)×U(N) bi-fundamental
	$V_{\bar\mu}^{(-1/2)}(y) = \pi(2\pi\alpha')^{\frac{3}{4}}g_0 \bar\mu^A \overline\Delta S_A e^{-\frac{1}{2}\phi}(y)$	
	$V_X^{(0)}(y) = 2\sqrt{2}\pi(2\pi\alpha')g_0 X_{\dot\alpha a}\Delta S^{\dot\alpha}\psi^a(y)$	
	$V_{\overline X}^{(0)}(y) = 2\sqrt{2}\pi(2\pi\alpha')g_0 \overline X_{\dot\alpha a}\overline\Delta S^{\dot\alpha}\psi^a(y)$	

Table 1. $\mathcal{N} = 4$ ADHM vertex operators

to find that the non-zero amplitudes in the limit $\alpha' \to 0$ are

$$\langle\langle V_Y^{(0)}V_\chi^{(-1)}V_{a'}^{(-1)}\rangle\rangle, \qquad \langle\langle V_X^{(0)}V_\chi^{(-1)}V_{\bar w}^{(-1)}\rangle\rangle, \qquad \langle\langle V_w^{(-1)}V_\chi^{(-1)}V_{\overline X}^{(0)}\rangle\rangle,$$
$$\langle\langle V_{\bar\mu}^{(-1/2)}V_\chi^{(-1)}V_\mu^{(-1/2)}\rangle\rangle, \langle\langle V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_\chi^{(-1)}\rangle\rangle, \langle\langle V_{\bar\psi}^{(-1/2)}V_{\bar\mu}^{(-1/2)}V_w^{(-1)}\rangle\rangle, \tag{2.2}$$
$$\langle\langle V_{\bar\psi}^{(-1/2)}V_{\bar w}^{(-1)}V_\mu^{(-1/2)}\rangle\rangle, \langle\langle V_{\bar\psi}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{a'}^{(-1)}\rangle\rangle, \langle\langle V_D^{(0)}V_{\bar w}^{(-1)}V_w^{(-1)}\rangle\rangle, \langle\langle V_D^{(0)}V_{a'}^{(-1)}V_{a'}^{(-1)}\rangle\rangle.$$

Calculating all the amplitudes and taking the limit, we find that the amplitudes in (2.2) are reproduced by the following low-energy effective action

$$S_{\text{str}}^0 = \frac{2\pi^2}{\kappa}\text{tr}_k\bigg[Y^m{}_a Y_{ma} - X_{\dot\alpha a}\overline X^{\dot\alpha}{}_a$$
$$+2Y^m{}_a[\chi_a, a'_m] - X_{\dot\alpha a}(\chi_a \bar w^{\dot\alpha} - \bar w^{\dot\alpha}\phi_a^0) - (w_{\dot\alpha}\chi_a - \phi_a^0 w_{\dot\alpha})\overline X^{\dot\alpha}{}_a$$
$$+\frac{1}{2}(\overline\Sigma^a)_{AB}\bar\mu^A(-\mu^B \chi_a + \phi_a^0 \mu^B) - \frac{1}{2}(\overline\Sigma^a)_{AB}\mathcal{M}'^{\alpha A}\mathcal{M}'_\alpha{}^B \chi_a$$
$$-i\bar\psi^{\dot\alpha}{}_A \left(\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}]\right)$$
$$-iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}\left(\bar w^{\dot\beta}w_{\dot\alpha} + \bar a'^{\dot\beta\alpha}a'_{\alpha\dot\alpha}\right)\bigg] \tag{2.3}$$

where ϕ_a^0 are the VEVs of the adjoint fields in $\mathcal{N} = 4$ super Yang-Mills theory. Note that the VEV-dependent amplitudes can be calculated similarly by replacing $\chi_a \to -\phi_a^0$.

The instanton effective action (2.3) is invariant under the supersymmetry transformations:

$$\delta a'_{\alpha\dot\alpha} = i\bar\xi_{\dot\alpha A}\mathcal{M}'_\alpha{}^A,$$
$$\delta\mathcal{M}'_\alpha{}^A = -2i\bar\xi_B^{\dot\alpha}(\Sigma^a)^{AB}Y_{ma}(\sigma^m)_{\alpha\dot\alpha},$$

$$\delta Y_{ma} = i\bar{\xi}_C^{\dot\beta}[\chi_b, \mathcal{M}'^{\alpha A}](\sigma^m)_{\alpha\dot\beta}(\bar\Sigma^{ba})_A{}^C + 2(\Sigma^a)^{AB}(\bar\sigma^{nm})^{\dot\gamma}{}_{\dot\beta}\bar\xi_B^{\dot\beta}[\bar\psi_{\dot\gamma A}, a'_n],$$

$$\delta w_{\dot\alpha} = i\bar\xi_{\dot\alpha A}\mu^A,$$

$$\delta\mu^A = 2i\bar\xi_B^{\dot\alpha}(\Sigma^a)^{AB}X_{\dot\alpha a},$$

$$\delta X_{\dot\alpha a} = 2i\bar\xi_{\dot\alpha B}(\bar\Sigma^{ba})_C{}^B(\mu^C\chi_b - \phi_b^0\mu^C) - \bar\xi_{\dot\beta A}(\Sigma^a)^{AB}(w_{\dot\alpha}\bar\psi_B^{\dot\beta} - 2\delta_{\dot\alpha}^{\dot\beta}w_{\dot\gamma}\bar\psi_B^{\dot\gamma}),$$

$$\delta\bar w_{\dot\alpha} = i\bar\xi_{\dot\alpha A}\bar\mu^A,$$

$$\delta\bar\mu^A = 2i\bar\xi_{\dot\alpha B}(\Sigma^a)^{AB}\bar X_a^{\dot\alpha},$$

$$\delta\bar X_a^{\dot\alpha} = 2i(\bar\Sigma^{ba})_A{}^C\bar\xi_C^{\dot\alpha}(\chi_b\bar\mu^A - \bar\mu^A\phi_b^0) - (\Sigma^a)^{AB}\bar\xi_{\dot\beta A}(\bar\psi_B^{\dot\beta}\bar w^{\dot\alpha} - 2\varepsilon^{\dot\alpha\dot\beta}\bar\psi_B^{\dot\gamma}\bar w_{\dot\gamma}),$$

$$\delta\chi_a = (\Sigma^a)^{AB}\bar\xi_{\dot\alpha A}\bar\psi_B^{\dot\alpha},$$

$$\delta\bar\psi_A^{\dot\alpha} = (\bar\Sigma^{ab})_A{}^B[\chi_a, \chi_b]\bar\xi_B^{\dot\alpha} - i\vec D\cdot\vec\tau^{\dot\alpha}{}_{\dot\beta}\bar\xi_A^{\dot\beta},$$

$$\delta\vec D = -i\vec\tau^{\dot\alpha}{}_{\dot\beta}(\Sigma^a)^{AB}\bar\xi_{\dot\alpha B}[\bar\psi_A^{\dot\beta}, \chi_a]. \tag{2.4}$$

After integrating out the auxiliary fields $Y_{ma}, X_{\dot\alpha a}, \bar X_{\dot\alpha a}$ in (2.3), we obtain the low-energy effective action [34]

$$\tilde S_{\rm str}^0 = \frac{2\pi^2}{\kappa}{\rm tr}_k\Big[-[\chi_a, a'_m]^2 + (w_{\dot\alpha}\chi_a - \phi_a^0 w_{\dot\alpha})(\chi_a\bar w^{\dot\alpha} - \bar w^{\dot\alpha}\phi_a^0)$$
$$+\frac{1}{2}(\overline\Sigma^a)_{AB}\bar\mu^A(-\mu^B\chi_a + \phi_a^0\mu^B) - \frac{1}{2}(\overline\Sigma^a)_{AB}\mathcal{M}'^{\alpha A}\mathcal{M}'_\alpha{}^B\chi_a\Big]$$
$$+S_{\rm ADHM}, \tag{2.5}$$

where

$$S_{\rm ADHM} = \frac{2\pi^2}{\kappa}{\rm tr}_k\Big[-i\bar\psi^{\dot\alpha}{}_A\big(\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}]\big)$$
$$-iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}\big(\bar w^{\dot\beta}w_{\dot\alpha} + \bar a'^{\dot\beta\alpha}a'_{\alpha\dot\alpha}\big)\Big], \tag{2.6}$$

is the terms providing the (fermionic) ADHM constraints. This action indeed agrees with the instanton effective action for $\mathcal{N} = 4$ super Yang-Mills theory based on the ADHM construction [35].

Let us introduce the (S,A)-type background. The vertex operator corresponding to this background in the $(-1/2, -1/2)$ picture is given as [25]

$$V_{\mathcal{F}}^{(-1/2,-1/2)}(z, \bar z) = (2\pi\alpha')\mathcal{F}^{(\alpha\beta)[AB]}S_\alpha S_A e^{-\frac{1}{2}\phi}(z)S_\beta S_B e^{-\frac{1}{2}\phi}(\bar z), \tag{2.7}$$

where we have identified the left- and right-moving fields due to the boundary condition at $z = \bar z$ [31]. Since the background contains two four-dimensional spin fields S_α, S_β, we need to insert other vertex operators including $S_\alpha S_\beta$ or ψ^m to get non-zero results. Otherwise the amplitude $\mathcal{A}$ behaves like $\mathcal{A} \propto \varepsilon_{\alpha\beta}\mathcal{F}^{(\alpha\beta)[AB]}$ and gives vanishing contribution. The candidates of such (combinations of) vertex operators are $V_{a'}, V_{\mathcal{M}'}V_{\mathcal{M}'}, V_D, V_Y$. However, we should also saturate the internal $SU(4)_R$ charge. It is impossible to do this only by the insertions of $V_{a'}, V_D$. Therefore non-zero amplitudes which contain one $V_{\mathcal{F}}^{(-1/2,-1/2)}$ must also contain $V_{\mathcal{M}'}V_{\mathcal{M}'}$ or V_Y. Considering power counting of α' and g_0, we find that the non-zero amplitudes after taking the zero-slope limit are

$$\langle\langle V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle, \qquad \langle\langle V_Y^{(0)}V_{a'}^{(-1)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle. \tag{2.8}$$

These amplitudes are evaluated in the appendix A. After taking the zero-slope limit, the first amplitude in (2.8) becomes

$$\langle\!\langle V_{\mathcal{M}'}^{(-1/2)} V_{\mathcal{M}'}^{(-1/2)} V_{\mathcal{F}}^{(-1/2,-1/2)} \rangle\!\rangle = \frac{2\pi^2}{\kappa} \mathrm{tr}_k \left[\frac{1}{2} (\overline{\Sigma}^a)_{AB} \mathcal{M}_\alpha'^A \mathcal{M}_\beta'^B (2\pi i^2)(2\pi\alpha')^{\frac{1}{2}} (\overline{\Sigma}^a)_{CD} \mathcal{F}^{(\alpha\beta)[CD]} \right].$$
(2.9)

The second amplitude in (2.8) is evaluated as

$$\langle\!\langle V_Y^{(0)} V_{a'}^{(0)} V_{\mathcal{F}}^{(-1/2,-1/2)} \rangle\!\rangle = \frac{2\pi^2}{\kappa} \mathrm{tr}_k \left[-\frac{i}{\sqrt{2}} (2\pi i)(\sigma^{mn})_{\alpha\beta} (\overline{\Sigma}^a)_{AB} Y_{ma} a_n' (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}^{(\alpha\beta)[AB]} \right].$$
(2.10)

These amplitudes are reproduced by the interaction terms on the D(-1)-branes induced by the (S,A)-type background, which are given by

$$\delta S_{(S,A)} = \frac{2\pi^2}{\kappa} \mathrm{tr}_k \left[2 Y_{ma} a_n' C^{mna} - \frac{1}{4} (\overline{\Sigma}^a)_{AB} \mathcal{M}_\alpha'^A \mathcal{M}_\beta'^B C^{(\alpha\beta)a} \right],$$
(2.11)

where

$$\begin{aligned}
C^{mna} &= \varepsilon_{\beta\gamma} (\sigma^{mn})_\alpha{}^\gamma (\bar{\Sigma}^a)_{AB} C^{(\alpha\beta)[AB]}, \\
C^{(\alpha\beta)a} &= (\bar{\Sigma}^a)_{AB} C^{(\alpha\beta)[AB]}, \\
C^{(\alpha\beta)[AB]} &= -2\pi (2\pi\alpha')^{\frac{1}{2}} \mathcal{F}^{(\alpha\beta)[AB]}.
\end{aligned}$$
(2.12)

We note that C^{mna} satisfies the self-dual condition

$$C^{mna} = \frac{1}{2} \varepsilon^{mnpq} C_{pq}{}^a.$$
(2.13)

The deformation term $\delta S_{(S,A)}$ is added to the undeformed part (2.5). After integrating out the auxiliary fields $Y_{ma}, X_{\dot\alpha a}$ and $\bar{X}_{\dot\alpha a}$, we obtain the following deformed instanton effective action

$$\begin{aligned}
\tilde{S}_{\mathrm{str}}^C = \frac{2\pi^2}{\kappa} \mathrm{tr}_k \Big[& - \left([\chi_a, a_m'] + C_{mna} a'^n\right)^2 + \left(w_{\dot\alpha}\chi_a - \phi_a^0 w_{\dot\alpha}\right)\left(\chi_a \bar{w}^{\dot\alpha} - \bar{w}^{\dot\alpha}\phi_a^0\right) \\
& + \frac{1}{2} (\overline{\Sigma}^a)_{AB} \bar{\mu}^A (-\mu^B \chi_a + \phi_a^0 \mu^B) - \frac{1}{2} (\overline{\Sigma}^a)_{AB} \mathcal{M}'^{\alpha A} \mathcal{M}_\alpha'{}^B \chi_a \\
& - \frac{1}{4} (\overline{\Sigma}^a)_{AB} C^{(\alpha\beta)a} \mathcal{M}_\alpha'{}^A \mathcal{M}_\beta'{}^B \Big] + S_{\mathrm{ADHM}}.
\end{aligned}$$
(2.14)

This is a natural $\mathcal{N} = 4$ extension of the $\mathcal{N} = 2$ deformed instanton effective action found in [22]. Note that at the second order in the R-R background, there is a mass term for the position moduli a_m' implying the fact that the position of the instantons are fixed at the origin of the D3-brane worldvolume.

The deformed instanton effective action (2.14) preserves half of the $\mathcal{N} = 4$ supersymmetry. We can see that the action is invariant under the following deformed supersymmetry transformations

$$\begin{aligned}
\delta a_{\alpha\dot\alpha}' &= i\bar{\xi}_{\dot\alpha A} \mathcal{M}_\alpha'{}^A, \\
\delta \mathcal{M}_\alpha'{}^A &= 2i\bar{\xi}_B^{\dot\alpha} (\Sigma^a)^{AB} (\sigma^m)_{\alpha\dot\alpha} ([\chi_a, a_m'] + a'^n C_{mna}),
\end{aligned}$$

$$\delta w_{\dot\alpha} = i\bar\xi_{\dot\alpha A}\mu^A,$$
$$\delta\mu^A = -2i\bar\xi^{\dot\alpha}_B(\Sigma^a)^{AB}(w_{\dot\alpha}\chi_a - \phi^0_a w_{\dot\alpha}),$$
$$\delta\bar w_{\dot\alpha} = i\bar\xi_{\dot\alpha A}\bar\mu^A,$$
$$\delta\bar\mu^A = -2i\bar\xi_{\dot\alpha B}(\Sigma^a)^{AB}(\chi_a\bar w^{\dot\alpha} - \bar w^{\dot\alpha}\phi^0_a),$$
$$\delta\chi_a = (\Sigma^a)^{AB}\bar\xi_{\dot\alpha A}\bar\psi^{\dot\alpha}_B,$$
$$\delta\bar\psi^{\dot\alpha}_A = (\bar\Sigma^{ab})_A{}^B[\chi_a,\chi_b]\bar\xi^{\dot\alpha}_B - i\vec D\cdot\vec\tau^{\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}_A,$$
$$\delta\vec D = -i\vec\tau^{\dot\alpha}{}_{\dot\beta}(\Sigma^a)^{AB}\bar\xi_{\dot\alpha B}[\bar\psi^{\dot\beta}_A,\chi_a]. \tag{2.15}$$

To show the invariance, the deformation parameters must satisfy the condition

$$C^{mn}{}_a C_{npb} - C^{mn}{}_b C_{npa} = 0. \tag{2.16}$$

As we will see in section 4, this corresponds to the flatness condition for the ten-dimensional Ω-background spacetime.

3 (S,A)-deformed $\mathcal{N} = 4$ super Yang-Mills theory

In this section, we calculate the instanton effective action from the (S,A)-deformed $\mathcal{N} = 4$ U(N) super Yang-Mills theory obtained in [25]. Since we are interested in obtaining instanton solutions, we perform the Wick rotation and consider the action in the Euclidean spacetime. The deformed Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_C, \tag{3.1}$$

where $\mathcal{L}_C$ denotes the (S,A)-deformation terms to the $\mathcal{N} = 4$ super Yang-Mills theory. $\mathcal{L}_0$ is the Lagrangian of $\mathcal{N} = 4$ U(N) super Yang-Mills theory

$$\mathcal{L}_0 = \frac{1}{\kappa}\text{Tr}\left[\frac{1}{4}F_{mn}F^{mn} + \frac{i\theta g^2}{32\pi^2}F_{mn}\tilde F^{mn} + \Lambda^{\alpha A}(\sigma^m)_{\alpha\dot\beta}D_m\bar\Lambda^{\dot\beta}_A + \frac{1}{2}(D_m\varphi_a)^2\right.$$
$$\left. -\frac{1}{2}g(\Sigma^a)^{AB}\bar\Lambda_{\dot\alpha A}[\varphi_a,\bar\Lambda^{\dot\alpha}_B] - \frac{1}{2}g(\bar\Sigma^a)_{AB}\Lambda^{\alpha A}[\varphi_a,\Lambda^B_\alpha] - \frac{1}{4}g^2[\varphi_a,\varphi_b]^2\right]. \tag{3.2}$$

Here $F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m,A_n]$ is the field strength of the U(N) gauge field A_m. $\tilde F^{mn} = \frac{1}{2}\varepsilon^{mnpq}F_{pq}$ is the dual field strength. $\Lambda^{\alpha A}, \bar\Lambda_{\dot\alpha A}$ are gauginos, φ_a are adjoint scalar fields, $D_m* = \partial_m * + ig[A_m,*]$ is the gauge covariant derivative, g is a gauge coupling constant and θ is a theta angle. We denote T^u as the basis of U(N) generators normalized as $\text{Tr}(T^u T^v) = \kappa\delta^{uv}$ with constant factor κ.[1] The (S,A)-deformation term $\mathcal{L}_C$ in (3.1) is given by [25]

$$\mathcal{L}_C = -\frac{1}{\kappa}\text{Tr}\left[igF_{mn}\varphi_a C^{mna} - g\varepsilon_{ABCD}\Lambda^A_\alpha\Lambda^B_\beta C^{(\alpha\beta)[CD]} + \frac{1}{2}g^2\varphi_a\varphi_b C_{mn}{}^a C^{mnb}\right] + \cdots, \tag{3.3}$$

where the deformation parameters C^{mna} and $C^{(\alpha\beta)[AB]}$ are defined in (2.12) and $\cdots$ stands for the $\mathcal{O}(C^3)$ contributions to the Lagrangian $\mathcal{L}$.

---

[1]We use the same normalization for U(N) generators and U(k) generators.

The equations of motion are

$$D^2\varphi_a - g(\Sigma^a)^{AB}\overline{\Lambda}_{\dot\alpha A}\overline{\Lambda}^{\dot\alpha}{}_B - g(\overline\Sigma^a)_{AB}\Lambda^{\alpha A}\Lambda_\alpha{}^B + g^2\Big[\varphi_b,[\varphi_a,\varphi_b]\Big] +$$
$$+ igF_{mn}C^{mna} + g^2\varphi^b C_{mn}{}^a C^{mnb} = 0,$$
$$(\sigma^m)_{\alpha\dot\beta}D_m\overline\Lambda^{\dot\beta}{}_A - g(\overline\Sigma^a)_{AB}[\varphi_a,\Lambda_\alpha{}^B] + 2g\varepsilon_{ABCD}\Lambda^{\beta B}C_{(\alpha\beta)}{}^{[CD]} = 0, \qquad (3.4)$$
$$(\bar\sigma^m)^{\dot\alpha\beta}D_m\Lambda_\beta{}^A - g(\Sigma^a)^{AB}[\varphi_a,\overline\Lambda^{\dot\alpha}{}_B] = 0,$$
$$D_m(F^{mn} + \tilde F^{mn} - 2ig\varphi_a C^{mna}) - ig[\varphi_a, D^n\varphi_a] - g(\sigma^n)_{\alpha\dot\beta}\{\Lambda^{\alpha A}, \overline\Lambda^{\dot\beta}{}_A\} = 0.$$

As in the case of the deformed $\mathcal{N}=2$ super Yang-Mills theory [27, 28], the terms which contain the gauge field strength and quadratic terms in C in the action are combined into the perfect square form S' as

$$S' = \int d^4x \frac{1}{\kappa}\text{Tr}\left[\frac{1}{2}(F_{mn}^{(-)})^2 - igC^{mna}\varphi_a F_{mn}^{(+)} - \frac{1}{2}g^2(C_{mna}\varphi_a)^2\right] + \left(\frac{8\pi^2}{g^2} + i\theta\right)k$$
$$= \int d^4x \frac{1}{\kappa}\text{Tr}\left[\frac{1}{2}\big(F_{mn}^{(+)} - igC_{mna}\varphi_a\big)^2\right] + \left(-\frac{8\pi^2}{g^2} + i\theta\right)k, \qquad (3.5)$$

where we have defined $F_{mn}^{(\pm)} = \frac{1}{2}(F_{mn} \pm \tilde F_{mn})$ and have used (2.13). We then obtain the self-dual and anti-self-dual equations for the gauge field

$$F_{mn}^{(-)} = 0, \qquad\qquad \text{for self-dual case,} \qquad\qquad (3.6)$$
$$F_{mn}^{(+)} - igC_{mna}\varphi_a = 0, \qquad\qquad \text{for anti-self-dual case.} \qquad\qquad (3.7)$$

In the Coulomb branch, the adjoint scalar fields φ_a are able to have VEVs. We need to expand the solution in the gauge coupling constant g and solve the equations perturbatively. However, unlike the $\mathcal{N}=2$ case, we can not solve the set of equations of motion exactly even for the $C=0$ case [35], so we will expand the field around the approximate solution of the equations. The expansion in g is valid when the VEVs of scalar fields are large. Then the classical action S is expanded in the gauge coupling constant g as

$$S = \frac{8\pi^2|k|}{g^2} + ik\theta + g^0 S_{\text{eff}}^{(0)} + \mathcal{O}(g^2), \qquad (3.8)$$

where $S_{\text{eff}}^{(0)}$ is the instanton effective action which is expressed by the ADHM moduli. To calculate the instanton effective action, we need to solve the equations of motion (3.4) in the instanton background at the leading order in the gauge coupling constant and write down the solution in terms of the ADHM moduli. Plugging this solution into the classical action, we obtain the instanton effective action $S_{\text{eff}}^{(0)}$.

Since the anti-self-dual solution is not deformed as we discuss later, we investigate the solutions for the self-dual case. For the self-dual condition (3.6), the solution is expanded in the gauge coupling as [35]

$$\begin{aligned}A_m &= g^{-1}A_m^{(0)} + g^1 A_m^{(1)} + \cdots, \\ \Lambda^A &= g^{-\frac{1}{2}}\Lambda^{(0)A} + g^{\frac{3}{2}}\Lambda^{(1)A} + \cdots, \\ \overline\Lambda_A &= g^{\frac{1}{2}}\overline\Lambda_A^{(0)} + g^{\frac{5}{2}}\overline\Lambda_A^{(1)} + \cdots, \\ \varphi_a &= g^0\varphi_a^{(0)} + g^2\varphi_a^{(1)} + \cdots.\end{aligned} \qquad (3.9)$$

Then the equations of motion in the self-dual background at the leading order are

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha}\nabla_m\Lambda_\alpha^{(0)A} = 0, \qquad (3.10)$$

$$(\sigma^m)_{\alpha\dot{\beta}}\nabla_m\overline{\Lambda}_A^{(0)\dot{\beta}} - i(\overline{\Sigma}^a)_{AB}\left[\varphi_a^{(0)},\Lambda_\alpha^{(0)B}\right] + 2\varepsilon_{ABCD}\Lambda^{(0)\beta B}C_{(\alpha\beta)}{}^{[CD]} = 0, \qquad (3.11)$$

$$\nabla^2\varphi_a^{(0)} - (\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}\Lambda_\alpha^{(0)B} + iF_{mn}^{(0)}C^{mna} = 0, \qquad (3.12)$$

$$\nabla_m(F^{(0)mn} + \tilde{F}^{(0)mn}) = 0, \qquad (3.13)$$

where ∇_m denotes the gauge covariant derivative in the self-dual instanton background. Similar to the $\mathcal{N}=2$ case [22, 27, 28], we can write the solution in the following form

$$
\begin{aligned}
A_m^{(0)} &= -i\overline{U}\partial_m U, \\
\Lambda_\alpha^{(0)A} &= \Lambda_\alpha(\mathcal{M}^A) = \overline{U}(\mathcal{M}^A f\bar{b}_\alpha - b_\alpha f\overline{\mathcal{M}}^A)U, \\
\varphi_a^{(0)} &= -\frac{1}{4}(\overline{\Sigma}^a)_{AB}\overline{U}\mathcal{M}^A f\overline{\mathcal{M}}^B U + \overline{U}\begin{pmatrix} \phi_a^0 & 0 \\ 0 & \chi_a\mathbf{1}_2 + \mathbf{1}_k C_a \end{pmatrix}U,
\end{aligned}
\qquad (3.14)
$$

where ϕ_a^0 are the VEVs of the adjoint scalar fields φ_a. C_a is the 2×2 matrix whose components are $(C_a)_\alpha{}^\beta = (\sigma^{mn})_\alpha{}^\beta C_{mna}$. χ_a should satisfy the equation

$$\mathbf{L}\chi_a = \frac{1}{4}(\overline{\Sigma}^a)_{AB}\bar{\mathcal{M}}^A\mathcal{M}^B + \bar{w}^{\dot{\alpha}}\phi_a^0 w_{\dot{\alpha}} + C^{mna}[a'_m, a'_n], \qquad (3.15)$$

where $\mathbf{L}$ is defined in (B.15). We present the derivation of these solutions in appendix B. As in the $\mathcal{N}=2$ case, the self-dual condition $F_{mn}^{(0)(-)} = 0$ is consistent with the equation of motion (3.13). Note that we do not need to find the solution for the anti-chiral fermion $\bar{\Lambda}^{(0)}$ since it contributes to the classical action as the subleading order in the gauge coupling constant.

Substituting the expansion (3.9) back into the classical action, the instanton effective action $S_{\text{eff}}^{(0)}$ in (3.8) is

$$
\begin{aligned}
S_{\text{eff}}^{(0)} = &\int d^4x \frac{1}{\kappa}\partial_m\text{tr}_k\left[\frac{1}{2}\varphi_a^{(0)}\nabla_m\varphi_a^{(0)}\right] \\
&+ \int d^4x \frac{1}{\kappa}\text{tr}_k\left[-\frac{1}{2}(\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}[\varphi_a^{(0)},\Lambda_\alpha^{(0)B}] - \frac{1}{2}(\overline{\Sigma}^a)_{AB}C^{(\alpha\beta)a}\Lambda_\alpha^{(0)A}\Lambda_\beta^{(0)B}\right. \\
&\left.+ \frac{1}{4}(\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}[\varphi_a^{(0)},\Lambda_\alpha^{(0)B}] - \frac{i}{2}\varphi_a^{(0)}F_{mn}^{(0)}C^{mna}\right],
\end{aligned}
\qquad (3.16)
$$

where we have decomposed the Yukawa term for later convenience. The first term in (3.16) is easily computed as in the case of the undeformed theory. After plugging the solution of the scalar field in (3.14) into the first term in the above expression, we find that the C-dependent parts are canceled out and the result is

$$\frac{2\pi^2}{\kappa}\text{tr}_k\left[\frac{1}{4}(\overline{\Sigma}^a)_{AB}\bar{\mu}^A\phi_a^0\mu^B - \bar{w}_{\dot{\alpha}}\phi_a^0\phi_a^0 w^{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\phi_a^0 w^{\dot{\alpha}}\chi_a\right]. \qquad (3.17)$$

Next, let us consider the second term in (3.16). To compute this term, we use the following relations

$$-\frac{1}{2}\Lambda^{(0)\alpha}(\mathcal{M}^A)[\varphi_a^{(0)},\Lambda_\alpha^{(0)}(\mathcal{M}^B)] = -\frac{1}{2}(\sigma^m)_{\alpha\dot\alpha}\Lambda^{(0)\alpha}(\mathcal{M}^A)\nabla_m\bar\psi^{\dot\alpha}{}_A - \frac{1}{2}\Lambda^{(0)\alpha}(\mathcal{M}^A)\Lambda_\alpha^{(0)}(\mathcal{N}_A)$$
$$-\frac{1}{2}\Lambda_\alpha^{(0)}(\mathcal{M}^A)\Xi^\alpha{}_A, \tag{3.18}$$

where

$$\bar\psi_A^{\dot\alpha} = \bar\psi_A^{(1)\dot\alpha} + \bar\psi_A^{(2)\dot\alpha} + \bar\psi_A^{(3)\dot\alpha},$$
$$\bar\psi_{\dot\alpha A}^{(1)} = \frac{1}{4}\varepsilon_{ABCD}\overline{U}\mathcal{M}^B f\overline{\Delta}_{\dot\alpha}\mathcal{M}^C f\overline{\mathcal{M}}^D U,$$
$$\bar\psi_{\dot\alpha A}^{(2)} = \frac{1}{2}(\overline{\Sigma}^a)_{AB}\overline{U}\left\{-\mathcal{M}^B f\overline{\Delta}_{\dot\alpha}M + M\Delta_{\dot\alpha}f\overline{\mathcal{M}}^B\right\}U, \tag{3.19}$$
$$\bar\psi_{\dot\alpha A}^{(3)} = \overline{U}Q_{\dot\alpha A}U,$$
$$\Xi_{\alpha A} = (\overline{\Sigma}^a)_{AB}(C^a)_\alpha{}^\beta\Lambda_\beta(\mathcal{M}^B),$$

and the matrices M, $Q_{\dot\alpha A}$ and $\mathcal{N}$ are given by

$$M_\lambda{}^\mu = M_{(u+i\alpha)}{}^{(v+j\beta)} = \begin{pmatrix} (\varphi_a^0)_u{}^v & 0 \\ 0 & (\chi_a)_i{}^j\delta_\alpha{}^\beta + \delta_i{}^j(C_a)_\alpha{}^\beta \end{pmatrix}, \tag{3.20}$$

$$Q_{\dot\alpha A} = \begin{pmatrix} 0 & 0 \\ 0 & (\mathcal{G}_{\dot\alpha A})_{ij}\delta_\alpha{}^\beta \end{pmatrix}, \tag{3.21}$$

$$\mathcal{N}_A = (\overline{\Sigma}^a)_{AB}\left[M\mathcal{M}^B - \mathcal{M}^B\chi_a\right] + 2\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{G}^{\dot\alpha}{}_A \end{pmatrix}a_{\dot\alpha} - 2a_{\dot\alpha}\mathcal{G}^{\dot\alpha}{}_A, \tag{3.22}$$

$$\overline{\mathcal{N}}_A = (\overline{\Sigma}^a)_{AB}\left[-\overline{\mathcal{M}}^B M + \chi_a\overline{\mathcal{M}}^B\right] + 2\bar a_{\dot\alpha}\begin{pmatrix} 0 & 0 \\ 0 & \overline{\mathcal{G}}^{\dot\alpha}{}_A \end{pmatrix} - 2\overline{\mathcal{G}}^{\dot\alpha}{}_A\bar a_{\dot\alpha}. \tag{3.23}$$

Here $\mathcal{G}_{\dot\alpha A}$ is a constant anti-Hermitian matrix and is determined so that $\mathcal{N}_A$ satisfies the fermionic ADHM constraint (B.10). The last term in (3.18) cancels the third term in (3.16) while the first term is rewritten as the total derivative term and does not contribute to the instanton effective action. The second term in (3.18) is evaluated by Corrigan's inner product formula [36], which is expressed as

$$\int d^4x \frac{1}{\kappa}\mathrm{tr}_k\left[-\frac{1}{2}\Lambda^{(0)\alpha}(\mathcal{M}^A)\Lambda_\alpha^{(0)}(\mathcal{N}_A)\right]$$
$$= \frac{2\pi^2}{\kappa}\mathrm{tr}_k\left[\frac{1}{2}(\overline{\Sigma}^a)_{AB}\left(\bar\mu^A\phi_a^0\mu^B - \bar\mu^A\mu^B\chi_a - \mathcal{M}'^{\alpha A}\mathcal{M}'_\alpha{}^B\chi_a\right) - \frac{1}{4}(\overline{\Sigma}^a)_{AB}C^{(\alpha\beta)a}\mathcal{M}'_\alpha{}^A\mathcal{M}'_\beta{}^B\right]. \tag{3.24}$$

Let us calculate the fourth and the last terms in the equation (3.16). To calculate these terms, we decompose the scalar field

$$\varphi_a^{(0)} = \varphi_{\mathcal{M},a}^{(0)} + \varphi_{\phi,a}^{(0)} + \varphi_{C,a}^{(0)}, \tag{3.25}$$

where

$$\varphi_{\mathcal{M},a}^{(0)} = -\frac{1}{4}(\overline{\Sigma}^a)_{AB}\overline{U}\mathcal{M}^A f\overline{\mathcal{M}}^B U + \overline{U}\begin{pmatrix} 0 & 0 \\ 0 & \chi_{\mathcal{M},a}\mathbf{1}_2 \end{pmatrix}U, \tag{3.26}$$

$$\varphi_{\phi,a}^{(0)} = \overline{U}\begin{pmatrix} \phi_a^0 & 0 \\ 0 & \chi_{\phi,a}\mathbf{1}_2 \end{pmatrix}U, \tag{3.27}$$

$$\varphi_{C,a}^{(0)} = \overline{U}\begin{pmatrix} 0 & 0 \\ 0 & \chi_{C,a}\mathbf{1}_2 + \mathbf{1}_k C_a \end{pmatrix}U. \tag{3.28}$$

We also decompose χ_a as

$$\chi_a = \chi_{\mathcal{M},a} + \chi_{\phi,a} + \chi_{C,a}\,, \tag{3.29}$$

where

$$\chi_{\mathcal{M},a} = \mathbf{L}^{-1}\left(\frac{1}{4}(\overline{\Sigma}^a)_{AB}\big(\bar{\mu}^A\mu^B + \mathcal{M}'^{\alpha A}\mathcal{M}'^{\,B}_\alpha\big)\right), \tag{3.30}$$

$$\chi_{\phi,a} = \mathbf{L}^{-1}(\bar{w}^{\dot\alpha}\phi_a^0 w_{\dot\alpha}), \tag{3.31}$$

$$\chi_{C,a} = \mathbf{L}^{-1}\left(C^{mna}[a'_m, a'_n]\right). \tag{3.32}$$

Then, we can rewrite the sum of the fourth and the last terms in (3.16) as

$$\int d^4x\, \frac{1}{\kappa}\mathrm{tr}_k\left[\frac{1}{4}(\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}[\varphi_{\mathcal{M},a}^{(0)} + \varphi_{\phi,a}^{(0)}, \Lambda_\alpha^{(0)B}] - \frac{1}{2}(\overline{\Sigma}^a)_{AB}\varphi_{C,a}^{(0)}\Lambda^{(0)\alpha A}\Lambda_\alpha^{(0)B}\right.$$
$$\left. -\frac{i}{2}C^{mna}\varphi_{\mathcal{M},a}^{(0)}F_{mn}^{(0)} - \frac{i}{2}C^{mna}(\varphi_{\phi,a}^{(0)} + \varphi_{C,a}^{(0)})F_{mn}^{(0)}\right]. \tag{3.33}$$

The first term in the above integral is independent of C and easily evaluated as

$$\frac{2\pi^2}{\kappa}\mathrm{tr}_k\left[-\frac{1}{4}(\overline{\Sigma}^a)_{AB}\left(\bar{\mu}^A\phi_a^0\mu^B - (\bar{\mu}^A\mu^B + \mathcal{M}'^{\alpha A}\mathcal{M}'^{\,B}_\alpha)(\chi_{\mathcal{M},a} + \chi_{\phi,a})\right)\right]. \tag{3.34}$$

Since $\varphi_{\mathcal{M},a}^{(0)}$ and $\varphi_{C,a}^{(0)}$ satisfy

$$\nabla^2\varphi_{\mathcal{M},a}^{(0)} = (\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}\Lambda_\alpha^{(0)B}, \tag{3.35}$$

$$\nabla^2\varphi_{C,a}^{(0)} = -iC^{mna}F_{mn}^{(0)}, \tag{3.36}$$

the second and third terms in the integral (3.33) become the total derivatives and are evaluated as

$$-\frac{1}{2}\lim_{|x|\to\infty}2\pi^2|x|^3\frac{x^m}{|x|}\mathrm{tr}_k\left[\varphi_{C,a}^{(0)}\nabla_m\varphi_{\mathcal{M},a}^{(0)} - \varphi_{\mathcal{M},a}^{(0)}\nabla_m\varphi_{C,a}^{(0)}\right]. \tag{3.37}$$

We find that this term vanishes at the boundary $|x| \to \infty$. Finally, we focus on the last term in (3.33). Since we have the relation

$$\varphi_{\phi,a}^{(0)} + \varphi_{C,a}^{(0)} = \overline{U}\begin{pmatrix} \phi_a^0 & 0 \\ 0 & (\chi_{\phi,a} + \chi_{C,a})\mathbf{1}_2 + \mathbf{1}_k C^a \end{pmatrix}U \equiv \overline{U}\hat{M}U, \tag{3.38}$$

the last term in (3.33) is rewritten as

$$-4(C^a)^\alpha{}_\beta \int d^4x \, \frac{1}{\kappa} \text{tr}_k \left[\overline{U} \hat{M} \mathcal{P} b_\alpha f \bar{b}^\beta U \right], \tag{3.39}$$

where $\mathcal{P}_\lambda{}^\mu$ is the projection operator defined by $\mathcal{P}_\lambda{}^\mu \equiv U_{\lambda u} \bar{U}_u{}^\mu = \delta^\mu{}_\lambda - \Delta_{\lambda i \dot\alpha} f_{ij} \bar{\Delta}_j{}^{\dot\alpha\mu}$. The term (3.39) can be evaluated as in the same way in $\mathcal{N} = 2$ case [27]. The result is

$$\frac{2\pi^2}{\kappa} \text{tr}_k \left[\frac{1}{4} C^{mna} C_{mna} \bar{w}^{\dot\alpha} w_{\dot\alpha} - C^{mna} (\chi_{\phi,a} + \chi_{C,a}) [a'_m, a'_n] \right]. \tag{3.40}$$

From these results $S_{\text{eff}}^{(0)}$ can be written in terms of ADHM moduli as follows

$$S_{\text{eff}}^{(0)} = \frac{2\pi^2}{\kappa} \text{tr}_k \left[\frac{1}{2} (\overline{\Sigma}^a)_{AB} \bar{\mu}^A \phi_a^0 \mu^B - \bar{w}_{\dot\alpha} \phi_a^0 \phi_a^0 w^{\dot\alpha} - \chi_a \mathbf{L} \chi_a \right.$$
$$\left. - \frac{1}{4} (\overline{\Sigma}^a)_{AB} C^{(\alpha\beta)a} \mathcal{M}'_\alpha{}^A \mathcal{M}'_\beta{}^B + \frac{1}{4} C^{mna} C_{mna} \bar{w}^{\dot\alpha} w_{\dot\alpha} \right]. \tag{3.41}$$

In the above instanton effective action, the $\mathcal{N} = 4$ ADHM moduli obey the ADHM constraints (B.4) and (B.10). We introduce auxiliary fields D^c and $\bar{\psi}_A^{\dot\alpha}$ for these constraints and add the Lagrange multiplier terms to the effective action. Then we can show that the $S_{\text{eff}}^{(0)}$ can be obtained from the following action by integrating out the auxiliary fields:

$$\tilde{S}_{\text{eff}}^{(0)} = \frac{2\pi^2}{\kappa} \text{tr}_k \left[-[\chi_a, a'_m]^2 + (w_{\dot\alpha} \chi_a - \phi_a^0 w_{\dot\alpha})(\chi_a \bar{w}^{\dot\alpha} - \bar{w}^{\dot\alpha} \phi_a^0) \right.$$
$$+ \frac{1}{2} (\overline{\Sigma}^a)_{AB} \bar{\mu}^A (-\chi_a \mu^B + \phi_a^0 \mu^B) - \frac{1}{2} (\overline{\Sigma}^a)_{AB} \mathcal{M}'^{\alpha A} \mathcal{M}'_\alpha{}^B \chi_a$$
$$\left. - \frac{1}{4} (\overline{\Sigma}^a)_{AB} C^{(\alpha\beta)a} \mathcal{M}'_\alpha{}^A \mathcal{M}'_\beta{}^B - 2 C^{mna} [a'_m, a'_n] \chi_a + \frac{1}{4} C^{mna} C_{mna} \bar{w}^{\dot\alpha} w_{\dot\alpha} \right]$$
$$+ S_{\text{ADHM}}. \tag{3.42}$$

We also call $\tilde{S}_{\text{eff}}^{(0)}$ the instanton effective action. The result does not agree with the string theory calculation (2.14) at the second order in the deformation parameter. This is the conceivable result since the (S,A)-deformed gauge theory does not break the translational invariance and the mass term for the position moduli a'_m is not allowed. To resolve this discrepancy, let us introduce the following term and improve the (S,A)-deformed theory:

$$\delta S = \frac{g^2}{2\kappa} \int d^4x \text{Tr} \left[C_{mpa} C_{nqa} x^p x^q F^{mr} F^n{}_r \right]. \tag{3.43}$$

This term does not provide any modifications to the equations of motion at the leading order in g and hence we can use the same solution (3.14) to calculate the instanton effective action. The additional contribution to the instanton effective action from the term (3.43) is easily evaluated as

$$\frac{g^2}{2\kappa} \int d^4x \text{Tr} \left[C_{mpa} C_{nqa} x^p x^q F^{(0)mr} F^{(0)n}{}_r \right] = \frac{2\pi^2}{\kappa} \text{tr}_k \left[-\frac{1}{4} (a'_m a'^m + \bar{w}^{\dot\alpha} w_{\dot\alpha}) \right] C^{pqa} C_{pqa}. \tag{3.44}$$

After adding this contribution to (3.42), the improved instanton effective action becomes the same as (2.14).

Before going to the next section, let us comment on the anti-self-dual case. For the anti-self-dual case, it is easy to see that these equations are not deformed by C. The instanton effective action still contains a term $F_{mn}^{(0)}\varphi_a^{(0)}C^{mna}$, but this term vanishes due to the self-dual condition of the background (2.13). Therefore the instanton effective action does not receive any deformation effect for the anti-self-dual case.

4 Ω-background deformation of $\mathcal{N} = 4$ super Yang-Mills theory

In this section, we discuss the relation between the deformed instanton effective action obtained in section 2, 3 and the one derived from $\mathcal{N} = 4$ super Yang-Mills theory in the Ω-background. We will find that the deformed instanton effective action (2.14) is interpreted as the one calculated in $\mathcal{N} = 4$ super Yang-Mills theory in the Ω-background. Similar to the Ω-background deformation of four-dimensional $\mathcal{N} = 2$ theories which is given by the dimensional reduction of the six-dimensional theory [23], the Ω-background deformation of $\mathcal{N} = 4$ theory can be obtained by the dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in the non-trivial metric

$$ds_{10}^2 = (dx^a)^2 + (dx^m + \Omega_{ma}dx^a)^2, \qquad \Omega_{ma} = \Omega_{mna}x^n. \tag{4.1}$$

Here $\Omega_{mna} = -\Omega_{nma}$. The four and six-dimensional indices m and a are raised and lowered by flat metric. The Lagrangian of $\mathcal{N} = 1$ super Yang-Mills theory in the metric (4.1) is given by

$$\mathcal{L}(\Omega) = \frac{1}{\kappa g^2}\mathrm{Tr}\sqrt{-g}\left[-\frac{1}{4}F_{\mathcal{M}\mathcal{N}}F_{\mathcal{P}\mathcal{Q}}g^{\mathcal{M}\mathcal{P}}g^{\mathcal{N}\mathcal{Q}} - \frac{i}{2}\overline{\Psi}e^{\mathcal{M}}{}_M\Gamma^M\mathcal{D}_{\mathcal{M}}\Psi\right], \tag{4.2}$$

where $F_{\mathcal{M}\mathcal{N}} = \partial_{\mathcal{M}}A_{\mathcal{N}} - \partial_{\mathcal{N}}A_{\mathcal{M}} + i[A_{\mathcal{M}}, A_{\mathcal{N}}]$ is the field strength of the gauge field $A_{\mathcal{M}}$ and Ψ is the ten-dimensional Majorana-Weyl spinor [37]. Γ^M is the ten-dimensional gamma matrix satisfying $\{\Gamma^M, \Gamma^N\} = -2\eta^{MN}$. $\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{Q} = 0, \ldots, 9$ are curved indices while $M, N, \ldots = 0, \ldots, 9$ are local Lorentz indices. $e^{\mathcal{M}}{}_M$ is the ten-dimensional vielbein and the covariant derivative is defined by

$$\mathcal{D}_{\mathcal{M}} = D_{\mathcal{M}} - \frac{1}{2}\omega_{\mathcal{M}MN}\Gamma^{MN}, \tag{4.3}$$

where $D_{\mathcal{M}}* = \partial_{\mathcal{M}} * + i[A_{\mathcal{M}}, *]$ and $\omega_{\mathcal{M}MN}$ is the spin connection. The Lorentz generator is defined as $\Gamma^{MN} = \frac{1}{4}[\Gamma^M, \Gamma^N]$. We require that the Ω-background spacetime is flat. As we will see later, this requirement ensures the existence of the supersymmetry for the instanton effective action. We find that the flatness condition of the spacetime needs

$$\Omega^{mn}{}_a\Omega_{npb} - \Omega^{mn}{}_b\Omega_{npa} = 0. \tag{4.4}$$

This is the natural generalization of the flatness condition in the six-dimensional Ω-background [23]. If the flatness condition (4.4) is satisfied, the only non-zero component in the spin connection for the metric (4.1) is given as

$$\omega_{\mathcal{A}mn} = -\Omega_{mn\mathcal{A}}. \tag{4.5}$$

After the dimensional reduction to four dimensions, we obtain the deformed $\mathcal{N} = 4$ super Yang-Mills theory. The action, which is expanded up to the second order in Ω_{mna}, has the form $\mathcal{L}(\Omega) = \mathcal{L}_0 + \delta\mathcal{L}(\Omega)$, where $\mathcal{L}_0$ is the Lagrangian of $\mathcal{N} = 4$ super Yang-Mills theory (3.2) and $\delta\mathcal{L}(\Omega)$ is the term which depends on the deformation parameter Ω_{mna}. This is given by

$$
\begin{aligned}
\delta\mathcal{L}(\Omega) = \frac{1}{\kappa}\mathrm{Tr}\Bigg[& gF_{mn}D^m\varphi_a\Omega^n{}_a + ig^2 D_m\varphi_a[\varphi_b,\varphi_a]\Omega^m{}_b + \frac{g^2}{2}F_{mp}F^p{}_n\Omega^m{}_a\Omega^n{}_a \\
& + \frac{g^2}{2}D_m\varphi_b D_n\varphi_a\Omega^m{}_a\Omega^n{}_b - \frac{ig^3}{2}F_{mn}[\varphi_a,\varphi_b]\Omega^m{}_a\Omega^n{}_b - \frac{g^2}{2}D_m\varphi_a D_n\varphi_a\Omega^m{}_b\Omega^n{}_b \\
& + \frac{ig}{2}\Omega^m{}_a\left[(\overline{\Sigma}^a)_{AB}\Lambda^{\alpha A}D_m\Lambda_\alpha{}^B + (\Sigma^a)^{AB}\overline{\Lambda}_{\dot\alpha A}D_m\overline{\Lambda}^{\dot\alpha}{}_B\right] \\
& - \frac{ig}{4}\Omega_{mna}\left[(\overline{\Sigma}^a)_{AB}\Lambda^{\alpha A}(\sigma^{mn})_\alpha{}^\beta\Lambda_\beta{}^B + (\Sigma^a)^{AB}\overline{\Lambda}_{\dot\alpha A}(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}\overline{\Lambda}^{\dot\beta}{}_B\right]\Bigg] + \mathcal{O}(\Omega^3). \quad (4.6)
\end{aligned}
$$

Here we have rescaled all the fields and the deformation parameter as $(A_m, \varphi_a, \Lambda_\alpha{}^A, \Omega_{ma}) \to g(A_m, \varphi_a, \Lambda_\alpha{}^A, \Omega_{ma})$ so that one can see clearly the power of the gauge coupling constant in each term.

Now we are interested in the instanton effective action of this deformed theory. Assuming the self-dual condition of the gauge field and using the gauge coupling expansion of the solution (3.9), the equations of motion at the leading order in g are given by

$$
i(\bar\sigma^m)^{\dot\alpha\alpha}\nabla_m\Lambda_\alpha^{(0)A} = 0,
$$
$$
i(\sigma^m)_{\alpha\dot\alpha}\nabla_m\overline{\Lambda}^{(0)\dot\alpha}{}_A + (\Sigma^a)_{AB}[\varphi_a^{(0)}, \Lambda_\alpha^{(0)B}] +
$$
$$
-i\Omega^m{}_a(\overline{\Sigma}^a)_{AB}\nabla_m\Lambda_\alpha^{(0)B} + \frac{i}{2}\Omega_{mn}{}^a(\overline{\Sigma}^a)_{AB}(\sigma^{mn})_\alpha{}^\beta\Lambda_\beta^{(0)B} = 0, \quad (4.7)
$$
$$
\nabla^2\varphi_a^{(0)} + F_{mn}^{(0)}\Omega^{mna} - (\nabla^m F_{mn}^{(0)})\Omega^{na} = (\overline{\Sigma}^a)_{AB}\Lambda_\alpha^{(0)A}\Lambda^{(0)\alpha B},
$$
$$
\nabla^m(F_{mn}^{(0)} + \tilde{F}_{mn}^{(0)}) = 0.
$$

If we identify the Ω-background and (S,A)-background parameters through the relation

$$
\Omega^{mna} = iC^{mna}, \quad (4.8)
$$

the equations of motion for $A_m^{(0)}, \varphi_a^{(0)}, \Lambda^{(0)}$ are precisely equivalent to the one in the (S,A)-deformed $\mathcal{N} = 4$ super Yang-Mills theory (3.4) because of the self-dual condition of the gauge field and the deformation parameters (2.13). On the other hand, the equation for $\overline{\Lambda}^{(0)}$ is different from the (S,A)-deformed theory. However, this does not matter when we discuss the instanton effective action because the contributions of $\bar\Lambda^{(0)}$ to the instanton effective action is subleading order in g and have no effects in the semi-classical approximation. Explicitly, after expanding the spacetime action around the instanton background, the terms at the order g^0 are given by

$$
\begin{aligned}
S_{\mathrm{eff}}^{(0)}(\Omega) = \int d^4x\, \frac{1}{\kappa}\mathrm{Tr}\Bigg[& \frac{1}{2}\nabla_m\varphi_a^{(0)}\nabla^m\varphi_a^{(0)} - \frac{1}{2}(\bar\Sigma^a)_{AB}\Lambda^{(0)\alpha A}[\varphi_a^{(0)}, \Lambda_\alpha^{(0)B}] - F_{mn}^{(0)}\varphi_a^{(0)}\Omega^{mna} \\
& - \frac{1}{2}F^{(0)mp}F_p^{(0)n}\Omega_m{}^a\Omega_n{}^a + \frac{i}{4}\Omega_{mna}(\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}(\sigma^{mn})_{\alpha\beta}\Lambda^{(0)\beta B}\Bigg]. \quad (4.9)
\end{aligned}
$$

There is no $\bar{\Lambda}^{(0)}$-dependence in (4.9). Here we have used

$$\Omega^m{}_a(\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}\nabla_m\Lambda^{(0)B}_\alpha = -\Omega^{mn}{}_a(\overline{\Sigma}^a)_{AB}\Lambda^{(0)\alpha A}(\sigma_{mn})_{\alpha\beta}\Lambda^{(0)\beta B}$$
$$+\text{(total derivative)}, \tag{4.10}$$

and the total derivative part does not contribute to $S^{(0)}_{\text{eff}}(\Omega)$ in the instanton background. Therefore, the instanton effective action for the $\mathcal{N} = 4$ super Yang-Mills theory in the Ω-background (4.9) is equivalent to the one for the $\mathcal{N} = 4$ (S,A)-deformed Yang-Mills theory (3.16) with the improvement term (3.43) under the identification (4.8).

5 (A,S)-deformed instanton effective action

In this section, we introduce the (A,S)-type background and study the deformed instanton effective action. In [25], we imposed the self-dual condition for the internal indices of the (A,S)-type background. In this paper we will consider the anti-self-dual condition, which induces the holomorphic deformation of the effective action. The vertex operator corresponding to this background in the $(-1/2, -1/2)$ picture is given as

$$V^{(-1/2,-1/2)}_{\mathcal{F}}(z,\bar{z}) = (2\pi\alpha')\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(AB)}S_{\dot\alpha}S^A e^{-\frac{1}{2}\phi}(z)S_{\dot\beta}S^B e^{-\frac{1}{2}\phi}(\bar{z}). \tag{5.1}$$

As in the case of the (S,A)-background, the non-zero amplitudes which include one (A,S)-background vertex operator are found to be

$$\langle\!\langle V^{(-1/2)}_{\mathcal{M}'}V^{(-1/2)}_{\mathcal{M}'}V^{(-1/2,-1/2)}_{\mathcal{F}}\rangle\!\rangle, \qquad \langle\!\langle V^{(-1/2)}_{\bar\mu}V^{(-1/2)}_{\mu}V^{(-1/2,-1/2)}_{\mathcal{F}}\rangle\!\rangle. \tag{5.2}$$

These amplitudes are evaluated in appendix A. After taking the zero-slope limit, these amplitudes are reproduced by the following interactions on the D(-1)-branes

$$\delta S_{(\text{A,S})} = -\frac{2\pi^2}{\kappa}\text{tr}_k\left[\left(2\bar\mu^A\mu^B + \mathcal{M}'^{\alpha A}\mathcal{M}'_\alpha{}^B\right)m_{(AB)}\right], \tag{5.3}$$

where we have defined the deformation parameter

$$m_{(AB)} \equiv \pi i(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(AB)}\varepsilon_{\dot\alpha\dot\beta}. \tag{5.4}$$

This result is also derived from the field theory side. The (A,S)-background induces new interaction terms on the D3-branes giving the deformed $\mathcal{N} = 4$ super Yang-Mills theory. After the Wick rotation the action of the (A,S)-deformed $\mathcal{N} = 4$ super Yang-Mills theory is [25]

$$\hat{S} = \int d^4x\,(\mathcal{L}_0 + \delta\mathcal{L}_{(\text{A,S})}), \tag{5.5}$$

where $\mathcal{L}_0$ is the Lagrangian of $\mathcal{N} = 4$ super Yang-Mills theory (3.2) and $\delta\mathcal{L}_{(\text{A,S})}$ is the induced interaction term given by

$$\delta\mathcal{L}_{(\text{A,S})} = \frac{1}{\kappa}\text{Tr}\left[-g^2(\Sigma^a\overline{\Sigma}^b\Sigma^c)^{AB}\varphi_a\varphi_b\varphi_c m_{(AB)} + 2g\Lambda^{\alpha A}\Lambda_\alpha{}^B m_{(AB)}\right.$$
$$\left. -\frac{1}{4}g^2(\Sigma^a\overline{\Sigma}^b\Sigma^c)^{AB}(\Sigma^a\overline{\Sigma}^b\Sigma^d)^{CD}\varphi_c\varphi_d m_{(AB)}m_{(CD)}\right]. \tag{5.6}$$

Since there are no gauge field interactions in (5.6), the (anti-)self-dual condition for the gauge fields $F_{mn}^{(\pm)} = 0$ is not modified by the (A,S)-background. As discussed in section 2, we expand the fields by the gauge coupling constant g and solve the equations of motion at the leading order in g. When we consider the anti-self-dual condition, we find that all the terms in (5.6) are subleading order in g. Therefore there are no (A,S)-background corrections to the instanton effective action for the anti-self-dual case. On the other hand, when we consider the self-dual case and expansion (3.9), we find that the bi-fermion term in (5.6) contributes to the instanton effective action while other terms are subleading order in g. This bi-fermion term is recognized as the mass term for the chiral fermion Λ. It is known that the chiral mass term is enough to see the effects of the instanton corrections to holomorphic quantities [35].

The equations of motion at the leading order for all the fields except $\bar{\Lambda}^{(0)}$ are the same with that of the undeformed $\mathcal{N} = 4$ super Yang-Mills theory. We can use the solution for the ordinary $\mathcal{N} = 4$ super Yang-Mills theory to compute the bi-fermion term. This term is evaluated by using Corrigan's inner product formula and the result is

$$\int d^4x \, \frac{2}{\kappa} \text{Tr} \left[\Lambda^{(0)\alpha A} \Lambda_\alpha^{(0)B} \right] m_{(AB)} = -\frac{2\pi^2}{\kappa} \text{tr}_k \left[2\bar{\mu}^A \mu^B + \mathcal{M}'^{\alpha A} \mathcal{M}'^B_\alpha \right] m_{(AB)}. \quad (5.7)$$

This completely agrees with the result of the string theory calculation (5.3) [38]. The instanton effective action for the (A,S)-deformed $\mathcal{N} = 4$ super Yang-Mills theory is therefore given by the sum of (2.3) and (5.7). In a suitable basis of $SU(4)_R$, the mass matrix is diagonalized as $m_{(AB)} = \text{diag}(m_1, m_2, m_3, m_4)$. If all the eigenvalues $m_1, \ldots, m_4$ are non-zero, the deformed instanton effective action corresponds to the one in massive deformation of $\mathcal{N} = 4$ super Yang-Mills theory. In particular, if two eigenvalues are zero and the others take the same non-zero value, the deformed instanton effective action is that of the mass deformed $\mathcal{N} = 2^*$ super Yang-Mills theory [38–40]. If one or three of the eigenvalues vanish and the others are non-zero, the deformed instanton effective action is that of the mass deformed $\mathcal{N} = 4$ super Yang-Mills theory which preserves $\mathcal{N} = 1$ supersymmetry [35].

The instanton effective action for the (A,S)-deformed $\mathcal{N} = 4$ super Yang-Mills theory is invariant under the supersymmetry transformation (2.4) with the following modifications for the transformations of $Y_{ma}, X_{\dot{\alpha}a}, \bar{X}_{\dot{\alpha}a}, \bar{\psi}_A^{\dot{\alpha}}$,

$$\begin{aligned}
\delta' Y_{ma} &= 2i\bar{\xi}^{\dot{\beta}}{}_C (\Sigma^a)^{AC} (\sigma^m)^\alpha{}_{\dot{\beta}} \mathcal{M}'_\alpha{}^B m_{(AB)}, \\
\delta' X_{\dot{\alpha}a} &= 4i\bar{\xi}_{\dot{\alpha}C} (\Sigma^a)^{AC} \mu^B m_{(AB)}, \\
\delta' \overline{X}^{\dot{\alpha}}{}_a &= -4i\bar{\xi}^{\dot{\alpha}}{}_C (\Sigma^a)^{AC} \bar{\mu}^B m_{(AB)}, \\
\delta' \bar{\psi}_A^{\dot{\alpha}} &= 4\bar{\xi}_C^{\dot{\alpha}} (\Sigma^a)^{BC} \chi_a m_{(AB)} - 4\bar{\xi}_C^{\dot{\alpha}} (\Sigma^a)^{BC} \phi_a^0 m_{(AB)}.
\end{aligned} \quad (5.8)$$

For these fields, the deformed supersymmetry transformation is given by $\delta + \delta'$ while for another fields, the supersymmetry transformation is not modified.

Finally, when we introduce the (S,A) and (A,S)-backgrounds simultaneously, the following cross term would be induced in the low-energy effective Lagrangian

$$\mathcal{L}_{\text{cross}} \sim \frac{1}{\kappa} \text{Tr} \left[g^2 [\varphi^a, \varphi^b] C_{mnc} (\Sigma^a \bar{\Sigma}^b \Sigma^c)^{AB} m_{AB} \right], \quad (5.9)$$

where the overall coefficient is determined through the explicit calculations of the open string disk amplitudes. This term is subleading order in g both in the self-dual and anti-self-dual cases. Therefore the instanton effective action for the self-dual case is simply given by the sum of (3.42) and (5.7).

6 Conclusions and discussion

In this paper, we investigated the instanton effective action of the $\mathcal{N} = 4$ super Yang-Mills theory deformed by the (S,A) and (A,S)-type backgrounds both from the string theory and field theory viewpoints.

In the string theory side, the instanton effective action is interpreted as the effective action of the D(−1)-branes in the D3-D(−1) system. This is evaluated from the disk amplitudes of open strings which have at least one endpoint attached on the D(−1)-branes. The R-R background is introduced by an insertion of the closed string vertex operators into the disk amplitudes.

In the case of (S,A)-type deformation, the moduli a'_m corresponding to the position of instantons get their mass at the second order of the background and hence these are fixed at the origin in the D3-brane worldvolume. This implies that the translational invariance is broken in the gauge theory.

On the other hand, from the viewpoint of the field theory, the instanton effective action is obtained through the ADHM construction of solutions in the (S,A)-deformed $\mathcal{N} = 4$ super Yang-Mills theory. Since this theory has the translational invariance, there is no mass term for a'_m in the instanton effective action. However, once we improve the spacetime action, the instanton effective action agrees with the string theory calculation.

We then compared the improved action for the (S,A)-deformed gauge theory with the one derived from the gauge theory in the Ω-background. The interaction terms contain explicit coordinate dependence and are quite different from the improved (S,A)-deformed theory. Nevertheless, we found that the equations of motion at the leading order in the self-dual gauge field background of the both theories coincide except for the one for the anti-chiral fermions. The instanton effective action in both theories also coincides. This is similar to the $\mathcal{N} = 2$ case [27, 28].

We also studied the effect of the (A,S)-type background. It can be interpreted as the mass term for the chiral fermion in the instanton effective action. Choosing the eigenvalues of the mass matrix appropriately, we obtained the instanton effective actions corresponding to massive deformations of $\mathcal{N} = 4$ super Yang-Mills theory with $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry. Thus the R-R 3-form backgrounds provide various deformations of field theories and instantons.

In this paper, we also showed the supersymmetry invariance of the deformed instanton effective action. It is important to study the topological symmetry of the deformed action since the BRST exactness of the instanton effective action is essential in the calculation of the instanton partition function. This subject will be discussed in a separate paper.

Recently, it is recognized that the deformation by the R-R 3-form background also plays an important role to investigate the heterotic-type I′ duality in study of eight-dimensional

exotic instantons [41, 42]. It would be interesting to study general deformed gauge theories and their non-perturbative effects in various dimensions and R-R backgrounds.

Acknowledgments

A   Detailed calculation of the disk amplitudes

In this appendix, we present the detailed calculations of the open string disk amplitudes including an insertion of the vertex operator corresponding to the closed string backgrounds. Our conventions and notations on the type IIB string worldsheet variables are found in [14, 17, 25]. We make use of the NSR formalism to calculate the amplitudes in ten-dimensional Euclidean spacetime. We are interested in open string disk amplitudes with at least one edge of the disk is attached on the D(-1)-branes. The open string vertex operators are inserted at the boundary of the disk parametrized by real coordinates y_i while the closed string vertex operator is inside the disk parametrized by a complex coordinate z. In general, the $(n + 2)$-point disk amplitude with n open string vertex operators $V_{\Phi_i}^{(q_i)}(y_i)$ and one closed string vertex operator $V_{\mathcal{F}}^{(-1/2,-1/2)}(z, \bar{z})$ with at least one boundary on the D(-1)-branes is given by

$$\langle\!\langle V_{\Phi_1}^{(q_1)} \cdots V_{\Phi_n}^{(q_n)} V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle = C_{-1} \int \frac{\prod_{i=1}^n dy_i dz d\bar{z}}{dV_{\mathrm{CKG}}} \langle V_{\Phi_1}^{(q_1)}(y_1) \cdots$$
$$\times V_{\Phi_n}^{(q_n)}(y_n) V_{\mathcal{F}}^{(-1/2,-1/2)}(z, \bar{z})\rangle, \qquad \text{(A.1)}$$

where $C_{-1} = \frac{1}{2\pi^2\alpha'^2}\frac{1}{\kappa g_0^2}$ is the normalization factor of the disk amplitudes. dV_{CKG} is an SL$(2, \mathbf{R})$ invariant volume factor of the conformal Killing group to fix three positions of the vertex operators. We fix $z \to i, \bar{z} \to -i$ and one of the y_i to ∞ in the following calculations. Note that because we are considering disk amplitudes, all the ϕ-charges in the bosonic ghost should add up to -2. In the following, we separately calculate the disk amplitudes with an insertion of the (S,A) and (A,S)-background.

A.1   (S,A)-background

As we mentioned in section 2, there are only two non-zero amplitudes containing one vertex operator of the (S,A)-background, which are given as

$$\langle\!\langle V_{\mathcal{M'}}^{(-1/2)} V_{\mathcal{M'}}^{(-1/2)} V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle, \quad \langle\!\langle V_Y^{(0)} V_{a'}^{(0)} V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle. \qquad \text{(A.2)}$$

The first amplitude is

$$\langle\langle V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle = \frac{1}{2\pi^2\alpha'^2}\frac{1}{\kappa g_0^2}(2\pi\alpha')^2 g_0^2(\pi^2)\mathrm{tr}_k\Big[\mathcal{M}'^{\alpha A}\mathcal{M}'^{\beta B}(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\gamma\delta)[CD]}\Big]$$

$$\times\int_{-\infty}^{y_1}dy_2\ (y_1-z)(y_1-\bar{z})(z-\bar{z})$$

$$\times\langle e^{-\frac{1}{2}\phi(y_1)}e^{-\frac{1}{2}\phi(y_2)}e^{-\frac{1}{2}\phi(z)}e^{-\frac{1}{2}\phi(\bar{z})}\rangle$$

$$\times\langle S_\alpha(y_1)S_\beta(y_2)S_\gamma(z)S_\delta(\bar{z})\rangle\langle S_A(y_1)S_B(y_2)S_C(z)S_D(\bar{z})\rangle.\ \text{(A.3)}$$

Here the correlators are calculated as

$$\langle e^{-\frac{1}{2}\phi(y_1)}e^{-\frac{1}{2}\phi(y_2)}e^{-\frac{1}{2}\phi(z)}e^{-\frac{1}{2}\phi(\bar{z})}\rangle = \Big[(y_1-y_2)(y_1-z)(y_1-\bar{z})(y_2-z)(y_2-\bar{z})(z-\bar{z})\Big]^{-\frac{1}{4}},$$

$$\langle S_\alpha(y_1)S_\beta(y_2)S_\gamma(z)S_\delta(\bar{z})\rangle = [(y_1-y_2)(y_1-z)(y_1-\bar{z})(y_2-z)(y_2-\bar{z})(z-\bar{z})]^{-\frac{1}{2}}$$

$$\times\{\varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}(y_1-y_2)(z-\bar{z})-\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}(y_1-\bar{z})(y_2-z)\}\,,\ \text{(A.4)}$$

$$\langle S_A(y_1)S_B(y_2)S_C(z)S_D(\bar{z})\rangle = \varepsilon_{ABCD}[(y_1-y_2)(y_1-z)(y_1-\bar{z})(y_2-z)(y_2-\bar{z})(z-\bar{z})]^{-\frac{1}{4}}\,.$$

$$\text{(A.5)}$$

We have omitted the overall cocycle factor [43] which will be multiplied at the end of calculations. After fixing the positions of vertex operators such as $y_1\to\infty, z\to i, \bar{z}\to -i$, and integrating over y_2, we get

$$\langle\langle V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle = \frac{2\pi^2}{\kappa}\mathrm{tr}_k\Big[\frac{1}{2}(\overline{\Sigma}^a)_{AB}\mathcal{M}'^A_\alpha$$

$$\times\mathcal{M}'^B_\beta(2\pi i^2)(2\pi\alpha')^{\frac{1}{2}}(\overline{\Sigma}^a)_{CD}\mathcal{F}^{(\alpha\beta)[CD]}\Big],\ \text{(A.6)}$$

where we have used the relation $\varepsilon_{ABCD}=\frac{1}{2}(\overline{\Sigma}^a)_{AB}(\overline{\Sigma}^a)_{CD}$ and multiplied the cocycle factor which has been evaluated as $+i$.

The second amplitude in (A.2) is

$$\langle\langle V_Y^{(0)}V_{a'}^{(-1)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\rangle = \frac{1}{2\pi^2\alpha'^2}\frac{1}{\kappa g_0^2}(2\pi\alpha')^2 g_0^2\left(\frac{4\pi^2}{\sqrt{2}}\right)\mathrm{tr}_k\Big[Y_{ma}a'_n(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}\Big]$$

$$\times\int_{-\infty}^{y_1}dy_2\ (y_1-z)(y_1-\bar{z})(z-\bar{z})\langle e^{-\phi(y_2)}e^{-\frac{1}{2}\phi(z)}e^{-\frac{1}{2}\phi(\bar{z})}\rangle$$

$$\times\langle\psi^m\psi^a(y_1)\psi^n(y_2)S_\alpha(z)S_A(z)S_\beta(\bar{z})S_B(\bar{z})\rangle.\ \text{(A.7)}$$

The correlator of the ghost fields is evaluated as

$$\langle e^{-\phi(y_2)}e^{-\frac{1}{2}\phi(z)}e^{-\frac{1}{2}\phi(\bar{z})}\rangle = (y_1-z)^{-\frac{1}{2}}(y_1-\bar{z})^{-\frac{1}{2}}(z-\bar{z})^{-\frac{1}{4}}.\ \text{(A.8)}$$

The correlator including spin fields is calculated as

$$\langle\psi^m\psi^a(y_1)\psi^n(y_2)S_\alpha(z)S_A(z)S_\beta(\bar{z})S_B(\bar{z})\rangle$$

$$= \frac{\delta^{mn}}{(y_1-y_2)}\langle\psi^a(y_2)S_\alpha(z)S_A(z)S_\beta(\bar{z})S_B(\bar{z})\rangle$$

$$+\frac{1}{2}(y_1-z)(\sigma^m)_{\alpha\dot{\alpha}}(\overline{\Sigma}^a)_{AC}\langle\psi^n(y_2)S^{\dot{\alpha}}(z)S^C(z)S_\beta(\bar{z})S_B(\bar{z})\rangle$$

$$+\frac{1}{2}(y_1-\bar{z})(\sigma^m)_{\beta\dot{\beta}}(\overline{\Sigma}^a)_{BD}\langle\psi^n(y_2)S_\alpha(z)S_A(z)S^{\dot{\beta}}(\bar{z})S^D(\bar{z})\rangle,\text{(A.9)}$$

where we have used the fact that $\psi^m\psi^a$ acts on the other fields in the correlator as the ten-dimensional Lorentz generator [43]. The first term in the above equation is proportional to $\varepsilon_{\alpha\beta}$ and will vanish when it is contracted with $\mathcal{F}^{(\alpha\beta)}$ while the second and the third terms are evaluated as

$$\frac{1}{2\sqrt{2}}(y_1-z)^{-1}\left[(y_2-z)^{-\frac{1}{2}}(y_2-\bar{z})^{-\frac{1}{2}}(z-\bar{z})^{-\frac{3}{4}}\right](\sigma^m)_{\alpha\dot{\alpha}}(\bar{\sigma}^n)^{\dot{\alpha}}{}_\beta(\overline{\Sigma}^a)_{AB}$$

$$+\frac{1}{2\sqrt{2}}(y_1-\bar{z})^{-1}\left[(y_2-z)^{-\frac{1}{2}}(y_2-\bar{z})^{-\frac{1}{2}}(z-\bar{z})^{-\frac{3}{4}}\right](\sigma^m)_{\beta\dot{\beta}}(\bar{\sigma}^n)^{\dot{\beta}}{}_\alpha(\overline{\Sigma}^a)_{BA}.\text{ (A.10)}$$

Here we have used the following relations

$$\langle\psi^n(y_2)S^{\dot{\alpha}}(z)S_\beta(\bar{z})\rangle = \frac{1}{\sqrt{2}}(\bar{\sigma}^n)^{\dot{\alpha}}{}_\beta(y_2-z)^{-\frac{1}{2}}(y_2-\bar{z})^{-\frac{1}{2}}, \tag{A.11}$$

$$\langle\psi^n(y_2)S_\alpha(z)S^{\dot{\beta}}(\bar{z})\rangle = \frac{1}{\sqrt{2}}(\sigma^n)_\alpha{}^{\dot{\beta}}(y_2-z)^{-\frac{1}{2}}(y_2-\bar{z})^{-\frac{1}{2}}, \tag{A.12}$$

$$\langle S_A(z)S^B(\bar{z})\rangle = \delta_A{}^B(z-\bar{z})^{-\frac{3}{4}}. \tag{A.13}$$

After evaluating the y_2 integration, the second amplitude in (A.2) is obtained as

$$\langle\!\langle V_Y^{(0)}V_{a'}^{(-1)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle=\frac{2\pi^2}{\kappa}\text{tr}_k\left[-\frac{i}{\sqrt{2}}(2\pi i)(\sigma^{mn})_{\alpha\beta}(\overline{\Sigma}^a)_{AB}Y_{ma}a'_n(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}\right],\text{(A.14)}$$

where we have multiplied the cocycle factor which has been calculated as $+i$.

A.2 (A,S)-background

The non-zero amplitudes including one (A,S)-background vertex operator are

$$\langle\!\langle V_{\bar{\mathcal{M}}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle, \qquad \langle\!\langle V_{\bar{\mu}}^{(-1/2)}V_{\mu}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle. \tag{A.15}$$

The first amplitude is

$$\langle\!\langle V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle = \frac{1}{2\pi^2\alpha'^2}\frac{1}{\kappa g_0^2}(2\pi\alpha')^2\pi^2g_0^2\text{tr}_k\left[\mathcal{M}'^{\alpha A}\mathcal{M}'^{\beta B}(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{[\dot{\alpha}\dot{\beta}]}{}_{(CD)}\right]$$

$$\times\int_{-\infty}^{y_1}dy_2\ (y_1-z)(y_1-\bar{z})(z-\bar{z})$$

$$\times\langle e^{-\frac{1}{2}\phi(y_1)}e^{-\frac{1}{2}\phi(y_2)}e^{-\frac{1}{2}\phi(z)}e^{-\frac{1}{2}\phi(\bar{z})}\rangle$$

$$\times\langle S_\alpha(y_1)S_\beta(y_2)\rangle\langle S_{\dot{\alpha}}(z)S_{\dot{\beta}}(\bar{z})\rangle$$

$$\times\langle S_A(y_1)S_B(y_2)S^C(z)S^D(\bar{z})\rangle. \tag{A.16}$$

The spin field correlators in the above equation are evaluated as

$$\langle S_\alpha(y_1)S_\beta(y_2)\rangle = \varepsilon_{\alpha\beta}(y_1-y_2)^{-\frac{1}{2}}, \tag{A.17}$$

$$\langle S_{\dot\alpha}(z)S_{\dot\beta}(\bar z)\rangle = \varepsilon_{\dot\alpha\dot\beta}(z-\bar z)^{-\frac12}, \tag{A.18}$$

$$\langle S_A(y_1)S_B(y_2)S^C(z)S^D(\bar z)\rangle = [(y_1-y_2)(y_1-z)(y_1-\bar z)(y_2-z)(y_2-\bar z)]^{-\frac34}(z-\bar z)^{-\frac14}$$
$$\times\left[-(y_1-\bar z)(y_2-z)\delta_A{}^C\delta_B{}^D + (y_1-z)(y_2-\bar z)\delta_A{}^D\delta_B{}^C\right]. \tag{A.19}$$

The correlator of the ghost part is given in (A.4). After performing the y_2 integration , the result is

$$\langle\!\langle V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{M}'}^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle = \frac{2\pi^2}{\kappa}\text{tr}_k\left[2\mathcal{M}'^{\alpha A}\mathcal{M}'_\alpha{}^B\pi i(2\pi\alpha')^{\frac12}\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(AB)}\varepsilon_{\dot\alpha\dot\beta}\right]. \tag{A.20}$$

Here the cocycle factor has been evaluated as $+1$.

The second amplitude is

$$\langle\!\langle V_{\bar\mu}^{(-1/2)}V_\mu^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle = \frac{1}{2\pi^2\alpha'^2}\frac{1}{\kappa g_0^2}(2\pi\alpha')^2 g_0^2\pi^2\left(\frac{2}{\sqrt{2}^2}\right)\text{tr}_k\left[\bar\mu^A\mu^B(2\pi\alpha')^{\frac12}\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(CD)}\right]$$
$$\times\int_{-\infty}^{y_1}dy_2\ (y_1-z)(y_2-\bar z)(z-\bar z)$$
$$\times\langle e^{-\frac12\phi(y_1)}e^{-\frac12\phi(y_2)}e^{-\frac12\phi(z)}e^{-\frac12\phi(\bar z)}\rangle$$
$$\times\langle\overline\Delta(y_1)\Delta(y_2)\rangle\langle S_{\dot\alpha}(z)S_{\dot\beta}(\bar z)\rangle$$
$$\times\langle S_A(y_1)S_B(y_2)S^C(z)S^D(\bar z)\rangle. \tag{A.21}$$

The correlators of the ghost and spin fields have been evaluated in (A.4), (A.18), (A.19). The correlator of the twist field is given as

$$\langle\bar\Delta(y_1)\Delta(y_2)\rangle = (y_1-y_2)^{-\frac12}. \tag{A.22}$$

After performing the y_2 integration, we find

$$\langle\!\langle V_{\bar\mu}^{(-1/2)}V_\mu^{(-1/2)}V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle = \frac{2\pi^2}{\kappa}\text{tr}_k\left[2\bar\mu^A\mu^B\pi i(2\pi\alpha')^{\frac12}\mathcal{F}^{[\dot\alpha\dot\beta]}{}_{(AB)}\varepsilon_{\dot\alpha\dot\beta}\right]. \tag{A.23}$$

Here the cocycle factor has been absorbed into the redefinition of μ and $\bar\mu$.

B   The ADHM construction

Here we briefly summarize the ADHM construction of instantons with topological number k, where k is a positive integer. We introduce the $(N+2k)\times 2k$ matrix $\Delta_{\lambda j\dot\alpha}$ which is given by

$$\Delta_{\lambda j\dot\alpha} = a_{\lambda j\dot\alpha} + b_{\lambda j}{}^\beta\sigma_{m\beta\dot\alpha}x^m, \tag{B.1}$$

where $\alpha,\dot\alpha = 1,2$, $\lambda = 1,2,\ldots,N+2k$ and $i,j = 1,2,\ldots,k$. $a_{\lambda j\dot\alpha}$ and $b_{\lambda j}{}^\beta$ are the constant matrices. They are decomposed as

$$a_{\lambda j\dot\alpha} = \begin{pmatrix} w_{uj\dot\alpha} \\ (a'_{\alpha\dot\alpha})_{ij} \end{pmatrix}, \quad b_{\lambda j}{}^\beta = \begin{pmatrix} 0 \\ \delta_{ij}\delta_\alpha{}^\beta \end{pmatrix}, \quad \lambda = u + i\alpha, \quad u = 1,2,\ldots,N. \tag{B.2}$$

These matrices w, a' are called bosonic ADHM moduli. Δ obeys the condition

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\beta}} = (f^{-1})_{ij}\delta^{\dot{\alpha}}{}_{\dot{\beta}}, \quad f_{ij} = \left[\frac{1}{2}\bar{w}_i^{\dot{\alpha}u}w_{uj\dot{\alpha}} + (x_m\delta_{ik}+(a'_m)_{ik})(x^m\delta_{kj}+(a'^m)_{kj})\right]^{-1}, \quad \text{(B.3)}$$

where the barred matrix denotes its Hermitian conjugate. The first equation in (B.3) is called ADHM constraint. In terms of the ADHM moduli a', w, and $\bar{w}$, the ADHM constraint can be rewritten as

$$(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{w}^{\dot{\beta}}w_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}}) = 0, \quad a'_m = \bar{a}'_m. \quad \text{(B.4)}$$

We also introduce $(N+2k)\times N$ matrix U which satisfies

$$\bar{\Delta}U = 0, \quad \bar{U}U = \mathbf{1}_N, \quad U\bar{U} + \Delta_{\dot{\alpha}}f\bar{\Delta}^{\dot{\alpha}} = \mathbf{1}_{N+2k}, \quad \text{(B.5)}$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. Then the self-dual equation $F_{mn}^{(0)(-)} = 0$ is solved in terms of U as

$$A_m^{(0)} = -i\bar{U}\partial_m U. \quad \text{(B.6)}$$

The corresponding gauge field strength $F_{mn}^{(0)}$ is given by

$$F_{mn}^{(0)} = -4i\bar{U}b^\alpha(\sigma_{mn})_\alpha{}^\beta f\bar{b}_\beta U. \quad \text{(B.7)}$$

We discuss the fermionic part in $\mathcal{N} = 4$ theory. We solve the Dirac equation $\bar{\sigma}^m\nabla_m\Lambda^{(0)A} = 0$, where ∇_m denotes the covariant derivative in the self-dual instanton background. The ansatz of the solution is

$$\Lambda_\alpha^{(0)A} = \bar{U}(\mathcal{M}^A f\bar{b}_\alpha - b_\alpha f\bar{\mathcal{M}}^A)U, \quad \text{(B.8)}$$

where $\mathcal{M}^A$ is the $(N+2k)\times k$ constant matrix. Plugging (B.8) into the Dirac equation, we obtain

$$\bar{\sigma}^m\nabla_m\Lambda^{(0)A} = 2\bar{U}b^\alpha f(\bar{\mathcal{M}}^A\Delta + \bar{\Delta}\mathcal{M}^A)f\bar{b}_\alpha U. \quad \text{(B.9)}$$

Then we have the fermionic ADHM constraint:

$$\bar{\mathcal{M}}^A\Delta + \bar{\Delta}\mathcal{M}^A = 0, \quad \text{(B.10)}$$

or equivalently

$$\bar{\mu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\mu^A + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot{\alpha}}] = 0, \quad \mathcal{M}'^A = \bar{\mathcal{M}}'^A, \quad \text{(B.11)}$$

where we have decomposed $\mathcal{M}^A$ as

$$\mathcal{M}_{\lambda j}^A = \begin{pmatrix} \mu_{uj}^A \\ (\mathcal{M}_\alpha'^A)_{ij} \end{pmatrix}, \quad \text{(B.12)}$$

μ_{uj}^A and $\mathcal{M}'^A$ are called the fermionic ADHM moduli.

Now we solve the equation of motion for the scalar field $\varphi_a^{(0)}$ (3.12) with the asymptotic boundary condition $\lim_{|x|\to\infty}\varphi_a^{(0)} = \phi_a^0$. First we consider the case of $C = 0$. The ansatz of the solution is

$$\varphi_a^{(0)} = -\frac{1}{4}(\bar{\Sigma}^a)_{AB}\bar{U}\mathcal{M}^A f\bar{\mathcal{M}}^B U + \bar{U}\begin{pmatrix} \phi_a^0 & 0 \\ 0 & \chi_a\mathbf{1}_2 \end{pmatrix}U. \quad \text{(B.13)}$$

Computing $\nabla^2 \varphi_a^{(0)}$ explicitly, we obtain

$$\nabla^2 \varphi_a^{(0)} = (\bar{\Sigma}^a)_{AB} \Lambda^{(0)A} \Lambda^{(0)B} + 4\bar{U}bf \left[\frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{\mathcal{M}}^A \mathcal{M}^B - \{f^{-1}, \chi_a\} + \bar{\Delta}^{\dot{\alpha}} \begin{pmatrix} \phi_a^0 & 0 \\ 0 & \chi_a \mathbf{1}_2 \end{pmatrix} \Delta_{\dot{\alpha}} \right] f\bar{b}U$$

$$= (\bar{\Sigma}^a)_{AB} \Lambda^{(0)A} \Lambda^{(0)B} + 4\bar{U}bf \left(\frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{\mathcal{M}}^A \mathcal{M}^B - \mathbf{L}\chi_a + \bar{w}^{\dot{\alpha}} \phi_a^0 w_{\dot{\alpha}} \right) f\bar{b}U, \quad \text{(B.14)}$$

where $\mathbf{L}\chi_a$ is defined by

$$\mathbf{L}\chi_a \equiv \frac{1}{2} \{\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}, \chi_a\} + \left[a'_m, [a'^m, \chi_a] \right]. \quad \text{(B.15)}$$

Then $\varphi_a^{(0)}$ satisfies the equation of motion if χ_a satisfies

$$\mathbf{L}\chi_a = \frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{\mathcal{M}}^A \mathcal{M}^B + \bar{w}^{\dot{\alpha}} \phi_a^0 w_{\dot{\alpha}}. \quad \text{(B.16)}$$

In the case of $C \neq 0$ we change the ansatz of the solution as

$$\varphi_a^{(0)} = -\frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{U} \mathcal{M}^A f \bar{\mathcal{M}}^B U + \bar{U} \begin{pmatrix} \phi_a^0 & 0 \\ 0 & \chi_a \mathbf{1}_2 + \mathbf{1}_k C_a \end{pmatrix} U, \quad \text{(B.17)}$$

where C_a is the 2×2 matrix of which components are $(C_a)_\alpha{}^\beta = (\sigma^{mn})_\alpha{}^\beta C_{mna}$. Now one can show that

$$\nabla^2 \varphi_a^{(0)} = (\bar{\Sigma}^a)_{AB} \Lambda^{(0)A} \Lambda^{(0)B} + 4\bar{U}bf \left[\frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{\mathcal{M}}^A \mathcal{M}^B \right.$$

$$\left. - 2f^{-1}C_a - \{f^{-1}, \chi_a\} + \bar{\Delta}^{\dot{\alpha}} \begin{pmatrix} \phi_a^0 & 0 \\ 0 & \chi_a \mathbf{1}_2 + \mathbf{1}_k C_a \end{pmatrix} \Delta_{\dot{\alpha}} \right] f\bar{b}U.$$

$$\text{(B.18)}$$

The second term in the square brackets in (B.18) becomes the deformation term $-iC^{mna} F_{mn}^{(0)}$ in the equation of motion by using (B.7). The C-dependent part in the last term in the square brackets becomes

$$\bar{\Delta}^{\dot{\alpha}} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_k C_a \end{pmatrix} \Delta_{\dot{\alpha}} = (\bar{a}' + x^m \bar{\sigma}_m)^{\dot{\alpha}\alpha} (C_a)_\alpha{}^\beta (a' + x^n \sigma_n)_{\beta\dot{\alpha}} = C^{mna} [a'_m, a'_n]. \quad \text{(B.19)}$$

Then we obtain

$$\nabla^2 \varphi_a^{(0)} = (\bar{\Sigma}^a)_{AB} \Lambda^{(0)A} \Lambda^{(0)B} - iC^{mna} F_{mn}^{(0)}$$

$$+ 4\bar{U}bf \left(\frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{\mathcal{M}}^A \mathcal{M}^B - \mathbf{L}\chi_a + \bar{w}^{\dot{\alpha}} \phi_a^0 w_{\dot{\alpha}} + C^{mna} [a'_m, a'_n] \right) f\bar{b}U. \quad \text{(B.20)}$$

Hence $\varphi_a^{(0)}$ is the solution of the deformed equation of motion if χ_a satisfies

$$\mathbf{L}\chi_a = \frac{1}{4} (\bar{\Sigma}^a)_{AB} \bar{\mathcal{M}}^A \mathcal{M}^B + \bar{w}^{\dot{\alpha}} \phi_a^0 w_{\dot{\alpha}} + C^{mna} [a'_m, a'_n]. \quad \text{(B.21)}$$

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